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CUTTING PLANE ALGORITHMS AND STATE SPACE CONSTRAINED LINEAR OPTIMAL CONTROL PROBLEMS

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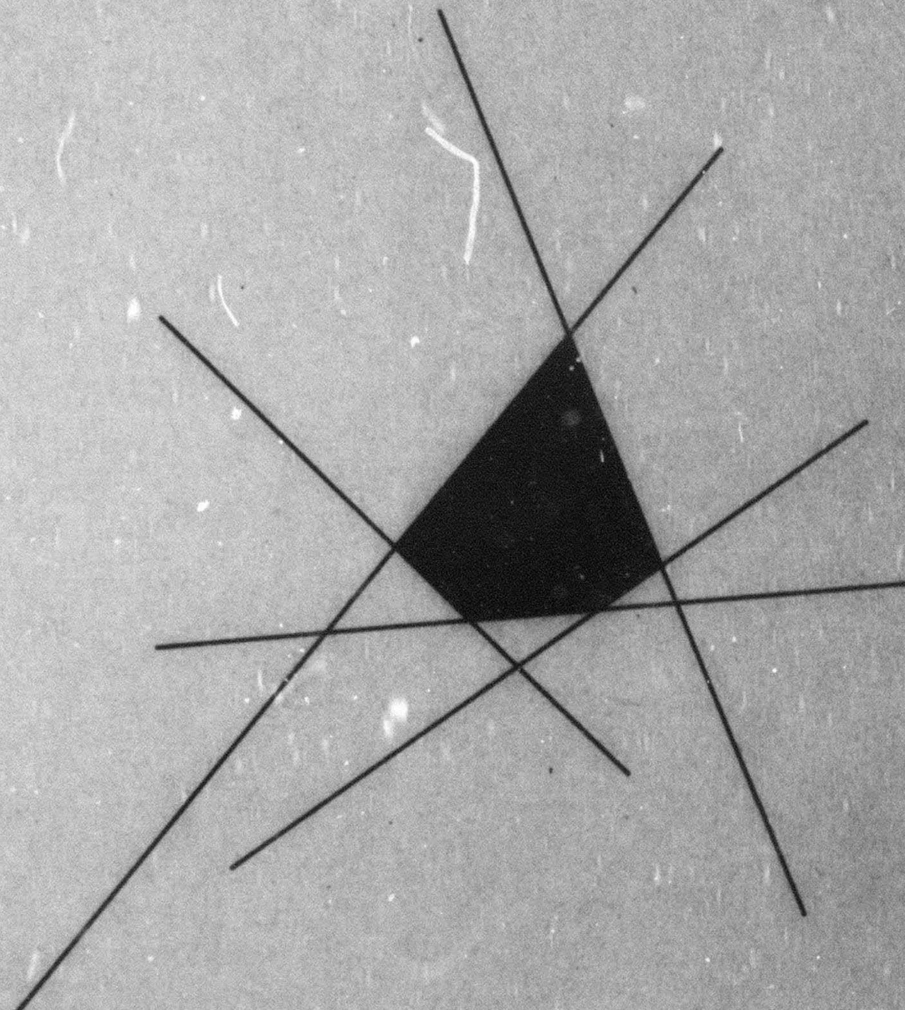

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CONSTRAINED LINEAR OPTIMAL CONTROL PROBLEMS

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ABSTRACT

In this paper an algorithm is proposed for solving continuous linear optimal control systems with state space constraints by solving a sequence of linear optimal control systems without state space constraints. The convergence of the algorithm is proved by method similar to cutting plane algorithm for convex programs in Banach Spaces. It is also shown how to solve the problem by using mathematical programming algorithm on the discretized problem. A numerical example is solved by discretization and mathematical programming.

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1. INTRODUCTION AND SUMMARY

The paper breaks naturally into three parts. The first part consists of Section 2 in which the Cheney-Goldstein, Kelley cutting plane algorithm for convex programming problems is reviewed and Section 3 where the algorithm is generalized to Banach Spaces. Theorem 3.1 is new, Theorem 3.2 is a special case of Theorem 10.2 of Levitin and Polyak [15], Theorem 3.3 is a slight generalization of a theorem due to Topkis [20].

The second part consists of Section 4 and 5 where a scheme for solving continuous linear optimal control problems with state space constraints by solving a sequence of problems without state space constraints is proposed. The convergence of the scheme is proved by identifying it as the cutting plane method of Section 3 applied to the optimal control problem. This scheme assumes one knows how to solve linear control problems without state space constraints and is relatively independent of what method for solving the problem without state space constraints is used. What happens to our proposal when the unconstrained problem is solved by using mathematical programming algorithms on the discretized problem is analyzed in the third part, Section 6. It is shown there, using a slight generalization of the results of J. Cullum [6], that as the mesh size of the discretization approaches zero the answers to the discrete problems converge on a subsequence to a solution of the original continuous problem. Finally, in Section 7 a numerical example is worked out.

2. THE CUTTING PLANE ALGORITHM IN FINITE DIMENSIONAL SPACES

Kelley [11] and Cheney-Goldstein [5] independently proposed a cutting plane algorithm for convex programming. The problem they consider is to

$$\text{(1)} \qquad \begin{aligned} &\underset{x}{\text{Minimize}} \quad cx \\ &\text{Subject to} \quad g(x) \le 0 \end{aligned}$$

where $c = (c_1, \ldots, c_n)$ and $x = (x_1, \ldots, x_n)$ are n vectors of real numbers and $g(x)$ is a real valued convex function. If g is differentiable at x^o with gradient $g'(x^o) = \left(\frac{\partial g}{\partial x_1}(x^o), \ldots, \frac{\partial g}{\partial x_n}(x^o)\right)$ then $g(x) \ge g(x^o) + g'(x^o)[x - x^o]$. Suppose now $x^o \notin K = \{x \mid g(x) \le 0\}$; i.e., $g(x^o) > 0$ then if $g(x) \le 0$ we must have

$$\text{(2)} \qquad g(x^o) + g'(x^o)[x - x^o] = g'(x^o)x + [g(x^o) - g'(x^o)x^o] \le 0 \ .$$

Notice that the above inequality, linear in x , is not satisfied for $x = x^o$, since $g(x^o) > 0$. If K is bounded this suggests the following algorithm. Since K is bounded we can find a matrix A and vector b so that $S = \{x \mid Ax \le b\}$ is compact and contains K . Then the algorithm is:

<u>Step 0</u>:

Solve the linear program

$$\text{(3)} \qquad \begin{aligned} &\text{Minimize} \quad cx \\ &\text{Subject to} \quad Ax \le b \ . \end{aligned}$$

Let x^o be an optimal solution to (3) and set $k = 0$.

<u>Step 1</u>:

If $g(x^k) \le 0$, x^k is an optimal solution to (1). If not add the inequality

$$g'(x^k)x \leq g'(x^k)x^k - g(x^k) \tag{4}$$

to (5) and go to Step 2.

<u>Step 2</u>:

Solve the linear program

$$\begin{aligned} &\text{Minimize} && cx \\ &\text{Subject to} && Ax \leq b \\ &&& g'(x^o)x \leq g'(x^o)x^o - g(x^o) \\ &&& \vdots \\ &&& g'(x^k)x \leq g'(x^k)x^k - g(x^k)\ . \end{aligned} \tag{5}$$

Let x^{k+1} be an optimal solution to (5). Set $k = k + 1$ and return to Step 1.

Figure 1 illustrates the algorithm for three steps.

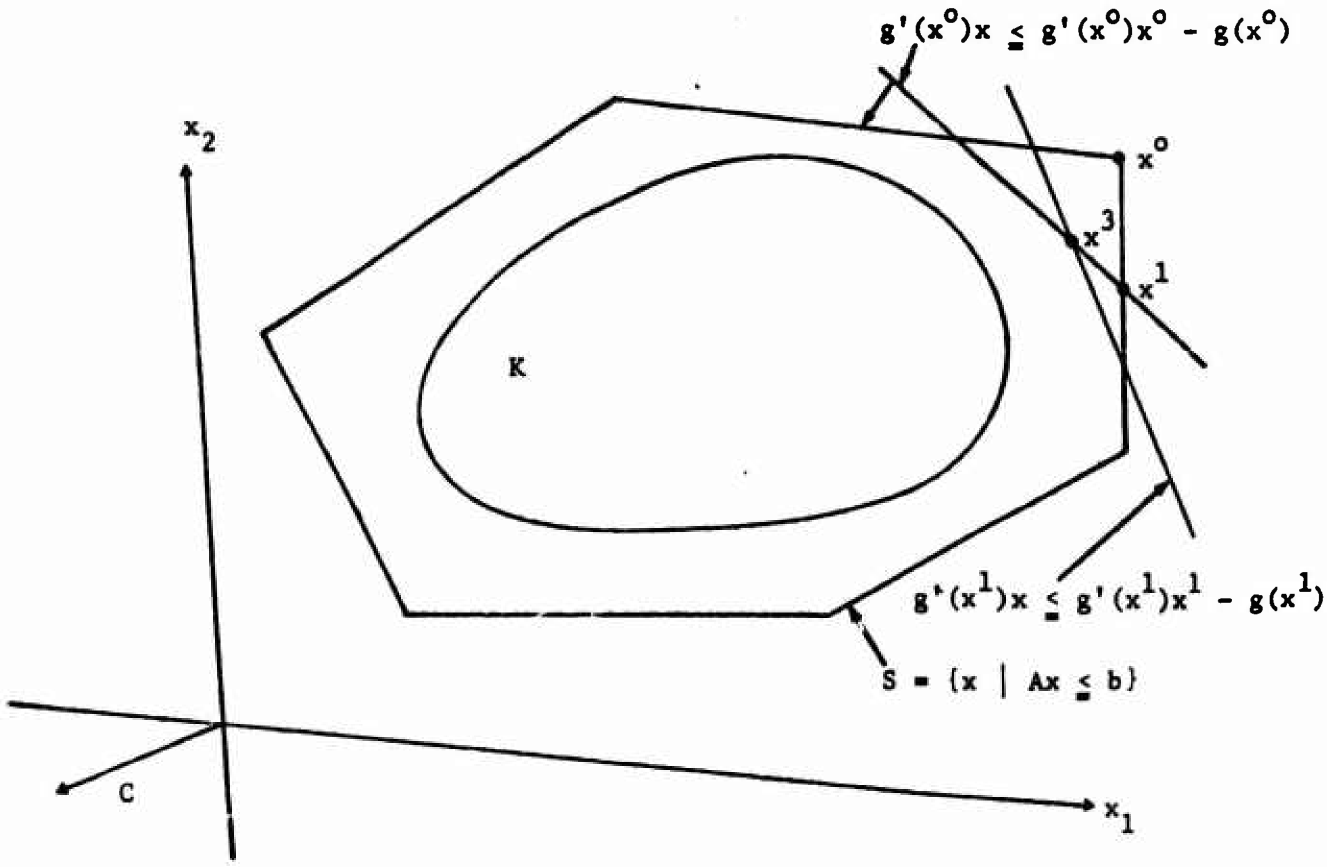


FIGURE 1

At each iteration at Step 2 we minimize cx over a set contained in S and containing K. At every iteration this set gets smaller so that in some sense x^k approaches K. More precisely we have:

Theorem 1 (Convergence):

Suppose in (1) that g is continuous and has uniformly bounded gradient on S; i.e., there exists $M > 0$ such that $||g'(x)|| \leq M$ for all $x \in S$. Then the sequence $\{x^k\}$ of points generated by the cutting plane algorithm has a convergent subsequence and for any convergent subsequence $\left\{x^{k_i}\right\}$ converging to say $\hat{x}$ we have that $\hat{x}$ solves (1).

Proof:

Since S is compact any sequence of points in S will have a convergent subsequence; in particular, the sequence $\{x^k\}$ is contained in S. Let $\left\{x^{k_i}\right\}$ be any convergent subsequence of the $\{x^k\}$, and let $\hat{z} = \operatorname{Inf}\{cx \mid g(x) \leq 0 , x \in S\}$. Since by (2) any x which satisfies $g(x) \leq 0$ satisfies the constraints of (5), $cx^{k_i} \leq \hat{z}$ for any i. Moreover $cx^{k_i} \leq cx^{k_{i+1}}$ since the constraint set of (5) gets no larger as k gets larger. Hence by the continuity of cx, $cx^{k_i} \to c\hat{x} \leq \hat{z}$. If $\hat{x} \in K$, i.e., $g(\hat{x}) \leq 0$ then $c\hat{x} \geq \hat{z}$ and hence $c\hat{x} = \hat{z}$ and $\hat{x}$ is an optimal solution. We now show that $\hat{x} \in K$. From (5) we have $g\left(x^{k_i}\right) \leq g'\left(x^{k_i}\right)\left[x^{k_i} - x^{k_j}\right]$ for any $j > i$. Letting both j and i go to infinity with $j > i$ we obtain

$$g(\hat{x}) \leq 0$$

since $\left[x^{k_j} - x^{k_i}\right]$ forms a Cauchy sequence converging to zero and $\left|\left|g'\left(x^{k_i}\right)\right|\right| \leq M$. In the next section we generalize this approach to function spaces.

3. A CUTTING PLANE ALGORITHM IN FUNCTION SPACE

The problem we consider here is

$$\text{(1)} \qquad \begin{array}{ll} \text{Minimize} & c(u) \\ \text{Subject to} & g(u) \leq 0 \\ & u \in S \subset \mathcal{U} \end{array}$$

where $\mathcal{U}$ is a Banach Space, $c(u)$ a lower semi-continuous convex functional on $\mathcal{U}$, and $g(u)$ a lower semi-continuous convex functional on $\mathcal{U}$. Finally S is convex and weakly sequentially compact.

Before we go through the details of the cutting plane algorithm for this problem it is perhaps worthwhile to consider the statement of (1) in a little more detail. Defined on a Banach Space $\mathcal{U}$ is a norm $||\cdot||$ such that $x^k \to x^o$ if and only if $||x^k - x^o|| \to 0$. Using this notion of convergence $g(u)$ is lower semi-continuous (ℓ.s.c.) on a set S if and only if $\{u \mid g(u) \leq \alpha\}$ is closed for all α. $g(u)$ is convex on a convex set S if and only if $\{(y,u) \mid y \geq g(u), u \in S\}$ is convex in $\mathcal{R} \times \mathcal{U}$.

Besides the convergence defined by the norm, usually called strong convergence, there is the notion of weak convergence. Associated with Banach Space $\mathcal{U}$ there is the adjoint space $\mathcal{U}^*$ which consists of all continuous linear functionals from $\mathcal{U}$ to the real line. Thus if $u^* \in \mathcal{U}^*$ we will denote by $\langle u^*,u\rangle$ the value of u^* at u. We say that a sequence $\{u^k\}$ converges weakly to u^o, written $u^k \overset{w}{\to} u^o$ if for each continuous linear functional $u^* \in \mathcal{U}^*$ $\langle u^*,u^k\rangle \to \langle u^*,u^o\rangle$. Armed with the notion of weak convergence we can define a set T to be weakly closed if $\{u^k\}$ is a sequence of T weakly convergent to u^o implies $u^o \in T$. Since strong convergence implies weak convergence a set which is weakly closed is *a posteriori* strongly closed but not necessarily the converse. We can then define $g(u)$ to be weakly lower semi-continuous (w.ℓ.s.c.) on S if $\{u \in S \mid g(u) \leq \alpha\}$ is weakly closed for all α. Clearly any continuous linear functional is weakly

continuous. It is an interesting property of convex functions that if they are strongly ℓ.s.c. they are w.ℓ.s.c. The proof depends on the following result from functional analysis.

__Lemma 1:__

A convex subset of a Banach space which is strongly closed is weakly closed.

__Proof:__

Dunford and Schwartz, p. 422, [10].

__Lemma 2:__

Let $g(u)$ be a ℓ.s.c. convex function on a closed convex subset of a Banach space $\mathcal{U}$. Then $g(u)$ is w.ℓ.s.c.

__Proof:__

$\{u \mid g(u) \leq c\}$ is closed for all c. $\{(y,u) \mid y \geq g(u)\ , u \in S\}$ is convex. Hence $\{u \mid g(u) \leq c\}$ is also convex. Therefore $\{u \mid g(u) \leq c\}$ is weakly closed for all c by Lemma 1. Hence $g(u)$ is w.ℓ.s.c.

Finally, we say that S is weakly (sequentially) compact if any sequence in S has a weakly convergent subsequence to a point in S.

One last point we need to generalize is the notion of a gradient. In our applications g will not necessarily be differentiable. A weaker concept called subdifferentiability suffices. Figure 2 illustrates the concept for a real valued convex function of one variable. The subgradient of

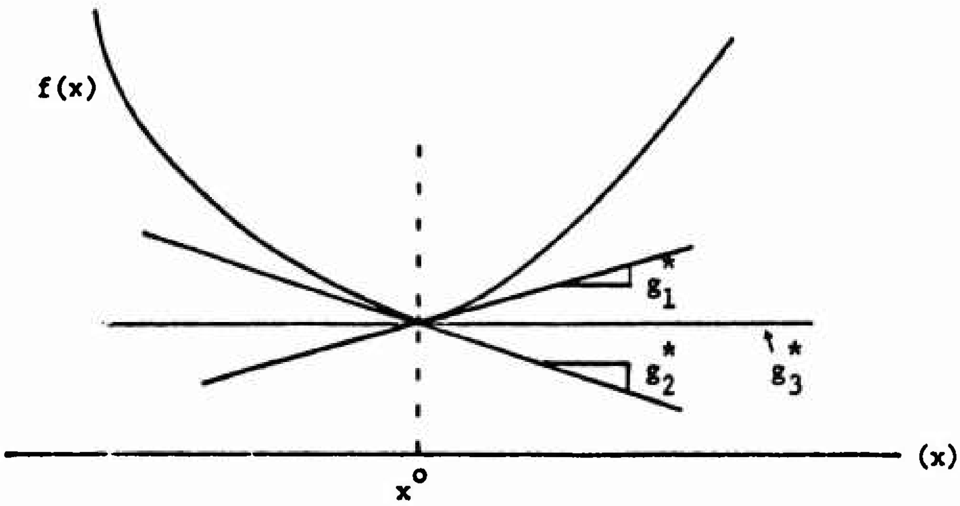


FIGURE 2

The slope of each of the indicated lines is a subderivative of f at x^o

g at u^o, $\partial g(u^o)$ is given by $\partial g(u^o) = \{g^* \varepsilon\, \mathcal{U}^* \mid g(u) \geq g(u^o) + \langle g^*, u - u^o\rangle$ for all $u \varepsilon\, \mathcal{U}\}$. If g has a gradient (Frechet derivative) $g'(u^o)$ at u^o then $\partial g(u^o) = \{g'(u^o)\}$ (Brønsted-Rockafellar [4]). We are now ready to discuss the generalized cutting plane algorithm.

<u>Step 0</u>:

We can solve

$$(2) \qquad \begin{array}{ll} \text{Minimize} & c(u) \\ \text{Subject to} & u \,\varepsilon\, S\ , \end{array}$$

since $c(u)$ is weakly lower semi-continuous by Lemma 2 and S is sequentially weakly compact. Call the solution u^o; let $k = 0$. Go to Step 1.

<u>Step 1</u>:

If $g(u^k) \leq 0$, u^k is the desired solution to (1). If not add the inequality

$$(3) \qquad \langle g^{*k}, u\rangle \leqq \langle g^{*k}, u^k\rangle - g(u^k)$$

to (4) and go to Step 2, where $g^{*k} \varepsilon \partial g(u^k)$. Go to Step 2.

Step 2:

Solve

$$(4) \qquad \begin{array}{ll} \text{Minimize} & c(u) \\ \text{Subject to} & u \varepsilon S \\ & \langle g^{*o}, u\rangle \leqq \langle g^{*o}, u^o\rangle - g(u^o) \\ & \vdots \\ & \langle g^{*k}, u\rangle \leqq \langle g^{*k}, u^k\rangle - g(u^k) \end{array}$$

obtaining as an optimal solution u^{k+1} , set $k = k + 1$ and go to Step 1.

Theorem 1:

Suppose in addition to the previous hypotheses for (1), we assume that $\left\{g^{*k}\right\}_{k=1}^{\infty}$ is compact in the strong topology and $g^{*k} \varepsilon \partial g(u^k)$ for each g^{*k} used in the cutting plane algorithm. Then the sequence $\{u^k\}$ of points generated by the algorithm contains a weakly convergent subsequence and for any weakly convergent subsequence $\left\{u^{k_i}\right\}$ converging to $\hat{u}$, $\hat{u}$ solves (1).

Proof:

Since S is weakly sequentially compact the sequence $\{u^k\}$ will have a weakly convergent subsequence. Let $\left\{u^{k_i}\right\}$ be a weakly convergent subsequence that converges to $\hat{u}$. Let $\hat{z} = \inf \{c(u) \mid g(u) \leqq 0 , u \varepsilon S\}$. For each i , $c\left(u^{k_i}\right) \leqq \hat{z}$ and hence $\limsup c\left(u^{k_i}\right) \leq \hat{z}$. Also, by the lower semi-continuity of $c(u)$, we have $\liminf c\left(u^{k_i}\right) \geq c(\hat{u})$. Hence

$c(\hat{u}) \leq \liminf c\left(u^{k_i}\right) \leq \limsup c\left(u^{k_i}\right) \leq \hat{z}$. Now we will show that

$$g(\hat{u}) \leq 0$$

and hence $c(\hat{u}) \geq \hat{z}$ so, $c(\hat{u}) = \hat{z}$.

By (4)

$$\langle g^{*k_i}, u^{k_j} \rangle \leq \langle g^{*k_i}, u^{k_i} \rangle - g\left(u^{k_i}\right)$$

or

$$g\left(u^{k_i}\right) \leq \langle g^{*k_i}, \left(u^{k_i} - u^{k_j}\right) \rangle \qquad \text{for } j > i .$$

Now we extract a subsequence ℓ_i from the sequence k_i such that $g^{*\ell_i} \to g^*$ strongly for some g^* ; this we can do by compactness. Let $v^i = u^{\ell_i} - u^{\ell_{i+1}}$, $v^i \overset{w}{\to} 0$ and $g\left(u^{\ell_i}\right) \leq \langle g^{*\ell_i}, v^i \rangle = \langle g^{*\ell_i} - g^*, v^i \rangle + \langle g^*, v^i \rangle$. Letting $i \to \infty$ we have

$$\lim_{i\to\infty} \langle g^*, v^i \rangle = 0 \quad \text{since} \quad v^i \overset{w}{\to} 0$$

$$\lim_{i\to\infty} |\langle g^{*\ell_i} - g^*, v^i \rangle| \leq \lim_{i\to\infty} ||g^{*\ell_i} - g^*|| \; ||v^i|| = 0$$

since $g^{*\ell_i} \to g^*$ strongly and $||v^i||$ is bounded [Dunford and Schwartz, p. 68]. On the other hand by weak lower semi-continuity

$$g(\hat{u}) \leq \liminf g\left(u^{\ell_i}\right) \leq \liminf \langle g^{*\ell_i}, u^{\ell_i} - u^{\ell_{i+1}} \rangle$$
$$= 0 .$$

Sometimes we can guarantee the strong convergence of the u^k rather than weak convergence on a subsequence.

Definition:

A convex function $c(u)$ defined on a convex subset K of a Banach Space $\mathcal{U}$ is *uniformly convex* if there exists a monotone function $\delta(\tau)$ on $\tau \varepsilon [0,\infty]$ with $\delta(\tau) > 0$ for $\tau > 0$ such that for all $u^1 \varepsilon K$, $u^2 \varepsilon K$ with $u^1 \neq u^2$ there exists $\lambda \varepsilon (0,1)$ such that

$$(1 - \lambda)c(u^1) + \lambda c(u^2) \geqq c((1 - \lambda)u^1 + \lambda u^2) + \delta(||u^1 - u^2||) .$$

Telser and Graves [18], Levitin and Polyak [15].

Theorem 2:

If in addition to the hypotheses of Theorem 1, $c(u)$ is uniformly convex then u^k converges strongly to u^o which corresponds to the unique optimal solution of (1).

Proof:

First we establish the uniqueness of the solution. Suppose u^1 and u^2 optimizes (1). Then $u^\lambda = (1 - \lambda)u^1 + \lambda u^2$ satisfies the constraints for all $\lambda \varepsilon (0,1)$ and if $u^1 \neq u^2$ by uniform convexity for some $\lambda \varepsilon (0,1)$ $c(u^\lambda) < (1 - \lambda)c(u^1) + \lambda c(u^2) = c(u^1) = c(u^2)$ but this contradicts the optimality of u^1 and u^2 so they must be the same point. Thus the optimal solution is unique. Now let u^k be the iterates defined by our algorithm. As $k \to \infty$ $c(u^k) \uparrow \bar{c}$ for some $\bar{c}$. If the original problem is feasible $\bar{c}$ is finite. For $\tau > 0$ let $\varepsilon = \delta(\tau) > 0$. Choose K sufficiently large so that for $n \geq k \geq K$, $c(u^n) - c(u^k) \leqq \varepsilon$. For any $n \geq k \geq K$ we can choose by uniform convexity $\lambda \varepsilon (0,1)$ such that

$$c(u^\lambda) = c((1 - \lambda)u^k + u^n) \leqq (1 - \lambda)c(u^k) + \lambda c(u^n) - \delta(||u^k - u^n||)$$

$$\leqq c(u^n) - \delta(||u^k - u^n||)$$

the last inequality resulting from the fact that $n \geq k$ implies $c(u^n) \geq c(u^k)$. On the other hand $c(u^k) \leq c(u^\lambda)$ since u^λ is feasible for the problem solved at the kth iteration. Thus $\delta(||u^k - u^n||) \leq c(u^n) - c(u^k) \leq \varepsilon$ which implies $||u^n - u^k|| \leq \tau$. Thus $||u^n - u^k||$ is a Cauchy Sequence and converges strongly to a limit. ||

Another consequence of uniform continuity is that we do not need to keep all the added constraints in the algorithm. This is important for computational efficiency. We revise our algorithm in the following way:

Revised Algorithm:

Step 0:

Solve

(5)
$$\begin{array}{ll} \text{Minimize} & c(u) \\ \text{Subject to} & u \in S \end{array}$$

for u^1, let $k = 1$. Go to Step 1.

Step 1:

If $g(u^k) \leq 0$, u^k is optimal. Stop. If not add the inequality constraint

(6)
$$\langle g^{*k}, u\rangle \leq \langle g^{*k}, u^k\rangle - g(u^k)$$

to the system in Step 2, where $g^{*k} \in \partial g(u^k)$. Go to Step 2.

Step 2:

Add k to L_{k-1} to obtain L_k, where L_k is the set of indices of the constraints added in Step 1 retained at the kth iteration.

$$\text{(7)} \qquad \begin{aligned} &\text{Minimize} \quad c(u) \\ &\text{Subject to} \quad u \in S \\ &\qquad \langle g^{*\ell}, u\rangle \leq \langle g^{*\ell}, u^{\ell}\rangle - g(u^{\ell})\ell \in L_k \end{aligned}$$

to obtain u^{k+1} . Delete all ℓ from L_k for which

$$\langle g^{*\ell}, u^{k+1}\rangle < \langle g^{*\ell}, u^{\ell}\rangle - g(u^{\ell}) .$$

Set $k = k + 1$. Go to Step 1.

This is the same as the previous algorithm except in Step 2 all the inequalities holding strictly for the optimal solution u^{k+1} are eliminated. We then have the following result due to Topkis [20]:

Theorem 3:

If the hypotheses of Theorem 2 are satisfied, the revised algorithm generates u^k which converge strongly to u^o which is the unique optimal solution for (1).

Proof:

$c(u^k) \leq c(u^{k+1})$ for all k thus $c(u^k) \to \bar{c}$ for some $\bar{c}$ which is finite if the original problem is feasible. Let c^* be the value of $c(u)$ for an optimal solution of (1). Then $\bar{c} \leq c^*$. Since S is weakly sequentially compact and all the $x^k \in S$, there is a subsequence x^{k_i} on which $x^{k_i} \underset{w}{\to} x^o$ for some $x^o \in S$. By the weak lower semi-continuity of $c(u)$ we have $c(u^o) \leq \bar{c} \leq c^*$. If we can show $g(u^o) \leq 0$ we have $c(u^o) \geq c^*$ and thus $c(u^o) = c^*$ and u^o is optimal. Let us now prove this. From the description of the revised algorithm it always follows that u^{k+1} satisfies the constraints of (7) at iteration k ; in particular we have:

$$\text{(8)} \qquad g\left(u^{k_i}\right) + g^*_{k_i}\left[u^{k_i+1} - u^{k_i}\right] \leq 0 .$$

By uniform convexity for each i there exists $\lambda_i \in (0,1)$ such that

$$(9) \qquad c\left[(1-\lambda_i)u^{k_i+1} + \lambda_i u^{k_i}\right] \leq (1-\lambda_i)c\left(u^{k_i+1}\right) + \lambda_i c\left(u^{k_i}\right) - \delta\left(||u^{k_i+1} - u^{k_i}||\right) .$$

But $(1-\lambda_i)u^{k_i+1} + \lambda_i u^{k_i}$ is feasible for (2) at iteration k_i thus

$$(10) \qquad c\left(u^{k_i}\right) \leq c\left[(1-\lambda_i)u^{k_i+1} + \lambda_i u^{k_i}\right] .$$

On the other hand $c\left(u^{k_i+1}\right) \geq c\left(u^{k_i}\right)$ so

$$(1-\lambda_i)c\left(u^{k_i+1}\right) + \lambda_i c\left(u^{k_i}\right) - \delta\left(||u^{k_i+1} - u^{k_i}||\right)$$

$$\leq c\left(u^{k_i+1}\right) - \delta\left(||u^{k_i+1} - u^{k_i}||\right) .$$

Thus from (9) and (10),

$$c\left(u^{k_i+1}\right) - c\left(u^{k_i}\right) \geq \delta\left(||u^{k_i+1} - u^{k_i}||\right) ,$$

so

$$(11) \qquad ||u^{k_i+1} - u^{k_i}|| \to 0 .$$

Thus from (6)

$$g\left(u^{k_i}\right) + ||\partial g\left(u^{k_i}\right)||\ ||u^{k_i+1} - u^{k_i}|| \leq 0 .$$

Letting $i \to \infty$ and observing that at least on a subsequence $||\partial g\left(u^{k_i}\right)||$ converges strongly to some number we have $g(u^o) \leq \underline{\lim}\ g\left(u^{k_i}\right) \leq 0$. Thus u^o is an optimal solution to (1). The proof of uniqueness is the same as for Theorem 2. To prove

strong convergence replace u^{k_i+1} by u^o starting with (9). The same step yields

(12) $$||u^o - u^{k_i}|| \to 0$$

which corresponds to (11).

4. OPTIMAL CONTROL PROBLEMS WITH STATE SPACE CONSTRAINTS

In this section we consider (1), in the next section various modifications and generalizations of this basic problem will be considered:

(1a)	Minimize	$C(u) = c(u,x(u))$	
(1b)	Subject to	$\dot{x}(t) = A(t)x(t) + B(t)u(t)$	a.e. $[0,T]$
(1c)		$x(0) = \xi$	
(1d)		$x(T) \in X^T$	
(1e)		$u(t) \in U(t)$	a.e. $[0,T]$
(1f)		$D(t)x(t) \leq d(t)$	a.e. $[0,T]$

where $x(t) = (x_1(t), \ldots, x_n(t))$ is the state vector, ξ is a given initial condition for x, $u(t) = (u_1(t), \ldots, u_r(t))$ is the control vector, $A(t)^{n\times n}$, $B(t)^{n\times r}$, $D(t)^{m\times n}$, $d(t)^{m\times 1}$ are matrix valued functions of $t \in [0,T]$ with the dimensions indicated, and $U(t)$ is a set valued function of time. $X^T \subset E^n$. $C(u) = c(u,x(u)) : L_2^r[0,T] \to E^1$ is convex and lower semi-continuous and $x(u)$ is the solution of (1b) corresponding to u. The initial assumptions we make on (1) are:

Assumptions:

Assumption A1:

$A(t)$, $B(t)$, $D(t)$, $d(t)$ are measurable on $[0,T]$ and uniformly bounded.

Assumption A2:

$u(t)$ is measurable on $[0,T]$.

Assumption A3:

$U(t)$ is compact and convex for each t and continuous in t with respect to the Hausdorff metric.

Assumption A4:

X^T is closed and convex.

First we will give the algorithm for (1) and then do the analysis which shows it is the cutting algorithm of Section 3 applied to (1). Since we are mainly interested in handling state space constraints we will assume we know how to solve the problems of the form (1) without state space constraints; namely, that we can solve the problem defined by (1a), (1b), (1c), (1d) and (1e). Methods for solving this later type of problem can be found in [2, 8, 9, 13]. The most natural of these from our point of view is the approach due to Dantzig [Dantzig, [9]], [Dantzig and Van Slyke, [8]]. These are somewhat difficult to apply if $c(u,x)$ depends on x explicitly.

Algorithm:

Step 0:

Find optimal solutions $u^1(t)$, $x^1(t)$ for the problem defined by (1a) to (1e). Set $k = 1$.

Step 1:

If $D(t)x^k(t) \leqq d(t)$ a.e. $[0,T]$ then the current solution is optimal. If not add x_{n+k} to the state vector where

$$(2) \qquad \frac{dx_{n+k}}{dt}(t) = [\pi(t;u^k)D(t)]x(t) - [\pi(t;u^k)d(t)]$$

where $x(t) = (x_1(t), \ldots, x_n(t))$ and where we impose the boundary conditions $x_{n+k}(0) = 0$, $x_{n+k}(T) \leqq 0$. $\pi(t,u^k) = (\pi_1, \ldots, \pi_n)$ is given by

$$(3) \qquad \pi_i(t,u^k) = \begin{cases} 1 & \text{if } D_i(t)x^k(t) - d_i(t) > 0 \\ 0 & \text{if } D_i(t)x^k(t) - d_i(t) \leqq 0 \end{cases}$$

for all $t \in [0,T]$ and $i = 1,2, \ldots, m$. Set $k = k + 1$.

Step 2:

Find optimal solutions $u^{k+1}(t)$, $x^{k+1}(t)$ for

(4a) Minimize $C(u)$

(4b) Subject to $\dot{x}(t) = A(t)x(t) + B(t)u(t)$ a.e. $[0,T]$

(4c) $\dot{x}_{n+j}(t) = [\pi(t;u^j)D(t)]x(t) - \pi(t;u^j)d(t)$ a.e. $[0,T]$, $j = 1, \ldots, k$.

(4d) $x(0) = \xi$, $x_{n+j}(0) = 0$ $j = 1, \ldots, k$.

(4e) $x(T) \in X^T$, $x_{n+j}(T) \leqq 0$ $j = 1, \ldots, k$.

(4d) $u(t) \in U(t)$ a.e. $[0,T]$.

Go to Step 1.

Note that at each iteration of the above algorithm we solve a problem without state space constraints. The cost of avoiding such constraints is that the state space vector increases by one dimension each iteration. The convergence properties of the algorithm is given in the following three theorems. They are proved by identifying the algorithm as a special case of the cutting plane algorithm described in Section 3.

Theorem 1:

Under Assumptions A1 to A4 the algorithm generates a sequence u^k, x^k which contains a subsequence u^{k_i}, x^{k_i} on which u^{k_i} converges weakly and x^{k_i} converges strongly. Moreover, for any subsequence such that $u^{k_i} \underset{w}{\to} u^o$, $x^{k_i} \to x^o$ we have that x^o, u^o solves (1).

Theorem 2:

If in addition to the hypotheses of Theorem 1, Assumption A5 is satisfied then

the iterates u^k , x^k given by the algorithm converge strongly to a unique solution u^o , x^o for (1).

Assumption A5:

$C(u)$ is uniformly convex in u for all u satisfying (1e).

Theorem 3:

Under the same hypotheses as Theorem 2 the iterates $\{u^k\}$ obtained from the following algorithm obtained by eliminating the strict inequalities converges strongly to the unique optimal solution of (1).

Revised Algorithm:

Step 0.

Solve the problem defined by (1a) to (1e) for $u^1(t)$, $x^1(t)$. Set $k = 1$. Go to Step 1.

Step 1:

If $D(t)x(t) \leq d(t)$ a.e. $[0,T]$ then the current solution is optimal. If not add x_{n+k} to the state vector (add $n + k$ to J_k where J_k is the set of indices of the state space co-ordinates retained at the kth iteration) where

$$\frac{dx_{n+k}}{dt}(t) = [\pi(t;u^k)D(t)]x(t) - [\pi(t;u^k)d(t)]$$

with boundary conditions $x_{n+k}(0) = 0$, $x_{n+k}(T) \leq 0$. $\pi(t;u^k)$ is given by (3). Go to Step 2.

Step 2:

Find optimal solutions $u^{k+1}(t)$, $x^{k+1}(t)$ for the problem defined by (4a), (4b), (4f) and

(5a) $$\dot{x}_{n+j} = [\pi(t;u^j)D(t)]x(t) - \pi(t;u^j)d(t) \quad \text{a.e. } [0,T] \quad j \in J_k$$

(5b) $$x(0) = \xi \ , \ x_{n+j}(0) = 0 \quad j \in J_k$$

(5c) $$x(T) \in X^T \ , \ x_{n+j}(T) \leq 0 \quad j \in J_k \ .$$

Delete all j from J_k for which $x_{n+j}^{k+1}(T) < 0$ to get J_{k+1}. Set $k = k + 1$. Return to Step 1.

A commonly occuring example of a $C(u)$ which is uniformly convex in u is the following

$$C(u) = c\left(\Phi(t;0)\xi + \int_0^t \Phi(t;s)B(s)u(s)ds, u\right)$$

where

$$c(x,u) = \int_0^T [x(t)Q(t)x(t) + u(t)P(t)u(t) + q(t)x(t) + p(t)u(t)]dt$$

and $Q(t)$ is symmetric and positive semi-definite for each $t \in [0,T]$, $P(t)$ is uniformly positive definite that is, $\inf_{||u(t)||=1} u(t)p(t)u(t) \geq \sigma > 0$ for all $t \in [0,T]$, and every component of $Q(t)$, $P(t)$, $q(t)$, $p(t)$ is square integrable on $[0,T]$.

Let us now identify the relations between the optimal control problems (1) and the generalized cutting plane method described in Section 3. To do this we first integrate (1b); this results in

(6) $$x(t) = \Phi(t;0)\xi + \int_0^t \Phi(t;s)B(s)u(s)ds$$

where $\Phi(t;s)$ is the matrix solution of

$$\frac{d}{dt}\Phi(t;s) = A(t)\Phi(t;s) \tag{7}$$

with the boundary condition

$$\Phi(s;s) = I \; . \tag{8}$$

This can always be carried out under A1 and A2. Moreover $\Phi(t;s)$ is absolutely continuous and uniformly bounded in s and t satisfying $0 \leq t \leq T$, $0 \leq s \leq T$ (Lee and Marcus [13]).

<u>Lemma 1</u>:

$\sup \{||u(t)|| \mid u(t) \in U(t) , t \in [0,T]\} < \infty$; i.e., the controls are uniformly bounded. Also

$$\sup \left\{ ||x(t)|| \mid x(t) = \Phi(t;0)\xi + \int_0^t \Phi(t;s)B(s)u(s)ds, t \in [0,T], u(t) \in U(t) \right\} < \infty \; .$$

<u>Proof</u>:

Suppose the lemma is false. Then there exists a sequence $u^k \in U(t_k)$ with $t_k \to t_o \in [0,T]$ and $||u^k|| \to +\infty$. But by the Hausdorff Continuity of $U(t)$ for any $\varepsilon > 0$ there is a $\delta > 0$ such that $|t_k - t_o| < \delta$ implies if $u^k \in U(t_k)$ then $\inf \{||u^k - u|| \mid u \in U(t_o)\} < \varepsilon$. But $\sup \{||u|| \mid u \in U(t_o)\}$ is finite so that $\limsup \{||U^k||\} \leq \sup \{||u|| \mid u \in U(t_o)\} + \varepsilon < \infty$.

$$||x(t)|| \leq ||\Phi(t;0)|| \; ||\xi|| + \int ||\Phi(t;s)|| \; ||B(s)|| \; ||u(s)||ds \; .$$

But all the norms on the right of the above expression are uniformly bounded so that $x(t)$ is uniformly bounded. ||

Let $S = \{u(t) \mid u(t) \in U(t)$ for $t \in [0,T]$ and $u(t)$ measurable$\}$.

Lemma 2:

Under A3, S is convex and weakly sequentially compact.

Proof:

S is bounded in $L_2^r[0,T]$ by Lemma 1. $L_2^r[0,T]$ is reflexive so that S is weakly sequentially conditionally compact by Theorem I.3.28 of Dunford and Schwartz [10]. S is obviously convex and strongly closed hence by Theorem V.3.13 of Dunford and Schwartz [10] S is weakly closed.

Let $z(t) = \Phi(t;0)\xi$ and define the operator $Y[u] : L_2^r[0,T] \to L_2^n[0,T]$ by

$$(9) \qquad Y[u] = y(t) = \int_0^t \Phi(t;s)B(s)u(s)ds \; .$$

Thus $x(t) = z(t) + y(t)$ where $y(t) = Y[u](t)$. The linear operator is *completely continuous* (or by some authors compact) if it maps bounded sets of the domain into conditionally strongly compact sets in the range (Liusternik and Sobolev, p. 129, [17]). The main consequence is for reflexive spaces weakly compact sets are mapped into compact sets (Dunford and Schwartz, p. 539, [10]).

Lemma 3:

Y is a completely continuous operator from $L_2^r[0,T]$ to $L_2^n[0,T]$.

For the proof of Lemma 3 one can use the example on p. 131 of Liusternik and Sobolev [17] by letting

$$K(s,t) = \begin{cases} \Phi(t;s)B(s) & 0 \le s \le t \le T \\ 0 \quad \text{otherwise} \; . \end{cases}$$

<u>Corollary</u>:

$Y[S] = \{Y[u] \mid u \in S\}$ is strongly compact and convex.

Let

$$g(u) = \int_0^T \pi(t;u)[D(t)x(t) - d(t)]dt$$

(10)

$$= \int_0^T \sum_{i=1}^{m} \text{Max}[0, D_i(t)x(t) - d_i(t)]dt$$

where $\pi(t,u)$ is defined by (3), and $D_i(t)$ and $d_i(t)$ are the ith rows of $D(t)$ and $d(t)$ respectively. Clearly $g(u) \leq 0$ if and only if (1f) is satisfied.

<u>Lemma 4</u>:

$g(u)$ is convex on S. It is strongly continuous on S and therefore weakly lower semi-continuous.

<u>Proof</u>:

Let $u^1 \varepsilon S$, $u^2 \varepsilon S$, $u^\lambda = (1-\lambda)u^1 + \lambda u^2$ for $0 < \lambda < 1$. Let x^1, x^2, and x^λ be the state trajectories resulting from u^1, u^2, and u^λ respectively. Then

$$(1-\lambda)g(u^1) + \lambda g(u^2) =$$

$$= \int_0^T \sum_{i=1}^m \left\{(1-\lambda)\text{Max}\left[0, D_i(t)x^1(t) - d_i(t)\right] + \lambda\text{Max}\left[0, D_i(t)x^2(t) - d_i(t)\right]\right\}dt$$

$$\geqq \int_0^T \sum_{i=1}^m \text{Max}\left\{0, (1-\lambda)\left(D_i(t)x^1(t) - d_i(t)\right) + \lambda\left(D_i(t)x^2(t) - d_i(t)\right)\right\}dt$$

$$= \int_0^T \sum_{i=1}^m \text{Max}\left\{0, D_i(t)x^\lambda(t) - d_i(t)\right\}dt$$

$$= g(u^\lambda) .$$

Thus g is convex on S . The operator which maps $u \to x = Y(u) + z$ is continuous from $L_2^r[0,T]$ to $L_2^n[0,T]$ because $Y(u)$ is completely continuous and therefore continuous. The mapping $x \to \delta_i(t) = \text{Max}[0, D_i(t)x(t) - d_i(t)]$ is a continuous mapping from $L_2^n[0,T]$ to $L_2^m[0,T]$ under A1. Thus

$$u(t) \to \sum_{i=1}^m \text{Max}[0, D_i(t)x(t) - d_i(t)]$$

is continuous. Finally $z \to \int_0^T 1 \cdot z(t)dt$ is in the conjugate space of $L_2^m[0,T]$ and is therefore continuous. Thus $g(u)$ is continuous. By Lemma 3.2 $g(u)$ is weakly lower semi-continuous.

To finish up the identification of the components of Theorem 3.1 we must define an element of the sub-differential of $g(u)$ at a point u^o . Let

$$\langle g^*(u^o), u\rangle = \int_C^T \pi(t;u^o)D(t)y(t)dt \tag{11}$$

where $y(t) = Y(u)$ and $\pi(t;u^o)$ is given by (3).

Lemma 5:

$g^*(u^o)$ given by (11) belongs to $\partial g(u^o)$ for all $u^o \in S$.

Proof:

We must show

$$(12) \qquad g(u) - g(u^o) \geqq \langle g^*(u^o), u - u^o \rangle$$

but

$$g(u) - g(u^o) = \int_0^T \{\pi(t;u)[D(t)[z(t) + y(t)] - d(t)]$$

$$- \pi(t;u^o)[D(t)[z(t) + y^o(t)] - d(t)]\}dt$$

$$\geqq \int_0^T \pi(t;u^o)\{[D(t)[z(t) + y(t)] - d(t)] - D(t)[z(t) + y^o(t)] - d(t)\}dt$$

$$= \int_0^T \pi(t;u^o)D(t)]y(t) - y^o(t)]dt$$

$$= \langle g^*(u^o), u - u^o \rangle$$

where $z(t) = \Phi(t;0)\xi$, $y = Y(u)$ and $y^o = Y(u^o)$. The inequality follows by (3) since $\pi_i(t;u^o) = 1$ if and only if $D_i(t)[z(t) + y^o(t)] - d_i(t) > 0$ thus

$$\pi(t;u^o)[D(t)[z(t) + y^o(t)] - d(t)] \geqq \pi(t;u)[D(t)[z(t) + y^o(t)] - d(t)] .$$

Finally:

Lemma 6:

$\{g^*(u^o) \mid u^o \in S\}$ is compact in the strong topology of $L_2^{r*}[0,T] = L_2^r[0,T]$.

Proof:

$\langle g^*(u^o),u\rangle$ can be written as $\langle g^*(u^o),u\rangle = \langle G^*(u^o),Y(u)\rangle$ where $G^*(u^o) : Y[s] \to E^1$ by the relation

$$(13) \qquad \langle G^*(u^o),y\rangle = \int_0^T \pi(t;u^o)D(t)y(t)dt \ .$$

Thus for each u^o , $G^*(u^o)$ is a continuous linear functional on $Y[S]$. $Y[S]$ is compact by the Corollary to Lemma 3. We will now use the Ascoli-Arzela Theorem Dunford and Schwartz, IV.6.7, [10] to show that the family $\mathcal{G} = \{G^*(u^o) \mid u^o \in S\}$ is compact in $C[Y[S]]$, the space of all continuous functions on $Y[S]$ to E^1 . First we show that $\mathcal{G}$ is uniformly bounded

$$|\langle G^*(u^o),y\rangle| = \left|\int_0^T \pi(t;u^o)D(t)y(t)dt\right|$$

$$\leqq \int_0^T \sqrt{m}\,\bar{D}\,\bar{y}\,dt = \sqrt{m}\,\bar{D}\,\bar{y}T$$

where $\bar{D} \geqq ||D(t)||$ for all t (A1) $\bar{y} \geqq ||y(t)||$ for all t and all $u \in S$ (Lemma 1) and $||\pi(t;u^o)|| \leqq \sqrt{m}$ (3). Now to show that the family is equi-continuous; that is, for all $\varepsilon > 0$ we must show there exists $\delta > 0$ such that if $||y^1 - y^2|| \leqq \delta$ then $|\langle G^*(u^o),y^1 - y^2\rangle| \leqq \varepsilon$ for all $u^o \in S$. But

$$|\langle G^*(u^o),y^1 - y^2\rangle| = \left|\int_0^T \pi(t;u^o)D(t)(y^1(t) - y^2(t))dt\right|$$

$$\leqq ||\pi(t;u^o)D(t)||\ ||y^1 - y^2||$$

$$\leqq \sqrt{m}\,\bar{D}\,\delta$$

where the norms are of $L_2^{n*}[0,T]$ and $L_2^n[0,T]$ respectively. Thus by the

Ascoli-Arzela Theorem the family G is compact in $C[Y[S]]$ and hence in $L_2^{n*}[0,T]$. Now we must show that the family $\{g^*(u^o) \mid u^o \in S\}$ is compact in L_2^{r*} . To do this we show for every sequence $u^k \in S$ there exists a convergent subsequence which we will also denote by u^k , $u^k \to u^o$ such that $\langle g^*(u^k) - g^*(u^o), u\rangle$ converges to 0 as $k \to \infty$ uniformly for all u such that $||u|| \leq 1$. But this is equivalent to

$$\langle G^*(u^k) - G^*(u^o), y(t)\rangle \to 0$$

uniformly for all $y \in Y(B)$ where B is the closed unit ball in $L_2^r[0,T]$. $Y[B]$ is compact by Lemma 3 and the fact $L_2^r[0,T]$ is reflexive and hence B is weakly sequentially compact Dunford and Schuartz, V.4.7, [10]. Thus in particular, $Y[B]$ is bounded. We choose as our subsequence one for which $G^*(u^k) \to G^*(u^o)$ strongly. Then $\sup \{\langle G^*(u^k) - G^*(u^o), y(t)\rangle \mid ||y(t)|| \leq K\} \to 0$ for any fixed K .

Proofs of Theorems 4.1, 4.2 and 4.3:

Theorems 4.1, 4.2 and 4.3 follow immediately from Theorems 3.1, 3.2 and 3.3 respectively under the following identifications:

Section 3	Section 4
$c(u)$	$C(u) = c(u,x(u))$
$g(u)$	$\int_0^T \pi(t,u)[D(t)x(t) - d(t)]dt$
S	$\{u(t) \mid u(t) \in U(t)$ for $t \in [0,T]$ and u is measurable$\}$
$\langle g^*(u^o),u\rangle$	$\int_0^T \pi(t,u^o)D(t)y(t)dt$.

For the last identification, at the kth iteration, we are adding the constraints

$$x_{n+k}(T) \leq 0$$

at the Step 2 in Section 4. We have $x_{n+k}(T) = \int_0^T \pi(t,u^k)D(t)y(t)dt$ + const. term and hence the addition of these constraints is equivalent to constraints added in Step 1 in the algorithm in Section 3.

5. MODIFICATIONS AND GENERALIZATIONS

Several generalizations and variations can be made to the basic algorithm. First we will discuss the question of variable initial points; that is suppose we replace the initial condition (4.1c) $x(0) = \xi$ by (1c') $x(t) \in X^o$. If $X^T = \{x^T\}$ is a single point then the same proofs work for convergence just by running time "backwards." In the more general case where neither X^o or X^T are singletons convergence can be proved where both X^o and X^T are closed convex sets and either X^o or X^T is bounded. We will not carry out the details here because the proof is rather messy. The main modification is in the resulting analog of Theorem 3.2 for uniformly convex objectives in u . Strong convergence of the trajectories in this case may occur only on a subsequence.

Other possible applications of the cutting plane algorithm to the state space controlled problem are also possible. These can be derived by changing $g(u)$ from the one defined by (4.10). Also more general constraints on the state than linear ones can be treated. As an example of other approaches the one in Levitin and Polyak, Section 10, [15] deserves mention. There the state space constraints (4.1f) are replaced by

$$q(x(t)) \leq 0 \qquad t \in [0,T]$$

where q is a real valued differentiable convex function of its argument. Then instead of the $g(u)$ defined by (4.10) they consider

$$g(u) = \max_{0 \leq t \leq T} q(x(t)) .$$

The cuts corresponding to (3.3) become

$$q\left(x^k(t_k)\right) + \langle q'\left(x^k(t_k)\right), x(t_k) - x^k(t_k)\rangle \leq 0$$

where x^k is the kth iterate and t_k is given by

$$g\left(x^k(t_k)\right) = \max_{0 \leq t \leq T} q(x^k(t)) \ .$$

However, we think our choice leads to computationally simpler algorithms.

Now it is probably proper to turn to the question of computation. Solving problem (4.4) by computer will inevitable require discretization at some point. In the next section we discretize from the beginning following Van Slyke and Wets [21]. We then use some results of Cullum [6] to show as the discretization mesh gets smaller and smaller the solution of the discretized problem approaches that of the continuous problem.

6. COMPUTATION BY DISCRETIZATION

Let us consider the problem

(1a) Minimize $C(u) = \int_0^T [c(u,t) + h(t)x(t)]dt$

(1b) Subject to $\dot{x}(t) = A(t)x(t) + B(t)u(t)$ a.e. $[0,T]$

(1c) $x(0) = \xi^o$

(1d) $x(T) = \xi^T$

(1e) $E(t)u(t) \leqq e(t)$

(1f) $D(t)x(t) \leqq d(t)$

where $c(u,t)$ is continuous and convex in u and measurable in t, $h(t)$ is measurable and uniformly bounded in t on $[0,T]$. $U(t) = \{u \mid E(t)u(t) \leqq e(t)\}$ is bounded for each t and as a set function is Hausdorff continuous, each element of $E(t)$ and $e(t)$ is in $L_2[0,T]$ and is uniformly bounded on $[0,T]$, we also assume $E(t)u(t) \leqq e(t)$ has a solution for each t. $A(t)$, $B(t)$ are continuous on $[0,T]$. This problem is a special case of (4.1). $E(t)^{p\times r}$, $e(t)^{p\times 1}$ are matrix valued functions of $t \in [0,T]$.

To discretize we first divide $[0,T]$ into K parts each of length $\Delta = T/K$. We then make the Cauchy-Euler approximation

(2) $$\frac{dx}{dt}(k\Delta) \cong \frac{x^{k+1} - x^k}{\Delta}$$

where we define $x^k = x(k\Delta)$, $k = 0,1, \ldots, K$. Many other approximations to the derivatives (Todd [19]) with a higher degree of accuracy can be analyzed using the same approach as the one presented here; however, the notation in these cases becomes extremely cumbersome. Then if we let $A^k = [I + \Delta A(k\Delta)]$, $B^k = \Delta B(k\Delta)$, $E^k = E(k\Delta)$, $e^k = e(k\Delta)$, $D^k = D(k\Delta)$, $d^k = d(k\Delta)$, $u^k = u(k\Delta)$, $c_k(u) = \Delta c(u,k\Delta)$, and $h^k(u) = h(u,k\Delta)$ (1) can be approximated by

$$\text{(3a)} \qquad \text{Minimize} \quad C(u) = \sum_{k=0}^{K-1} c_k(u^k) + h^k x^k$$

$$\text{(3b)} \qquad \text{Subject to} \quad x^{k+1} = A^k x^k + B^k u^k \qquad k = 0,1, \ldots, K-1$$

$$\text{(3c)} \qquad x^o = \xi^o$$

$$\text{(3d)} \qquad x^K = \xi^K$$

$$\text{(3e)} \qquad E^k u^k \leqq e^k \qquad k = 0,1, \ldots, K-1$$

$$\text{(3f)} \qquad D^k x^k \leqq d^k \qquad k = 1, \ldots, K-1 .$$

This is a mathematical programming problem with linear constraints and a convex objective function for which many algorithms exist. See for example, Abadie [1], Cheney-Goldstein [5], Kelley [11], Künzi and Krelle [12], Zoutendijk [22]. However, (3) has very special structures which allows much simplification [Dantzig and Van Slyke, 1969, Section 4]. To see that this is necessary let us consider the number of constraints and variables in (3). There are Kn equations in (3b), n equations in (3c), n equations in (3d), $K \times p$ inequalities in (3e), and $(K - 1) \times \ell$ inequalities associated with (3f) where ℓ is the number of rows of D^k . This adds up to $K[n + m + \ell + p] - \ell + 2n$ equations in $nK + rK$ variables. The parameters n , ℓ , m and p are usually relatively small; however, K determines the accuracy of the approximation (2) and could be extremely large if a high degree of accuracy is required. In order to reduce the problem we start by analyzing a much simpler problem than (3) and then solve a sequence of these easier problems in order to solve (3). Suppose we consider (3) without the state space constraints (3f) then the problem can be simplified considerably. Let

$$\text{(4)} \qquad \begin{aligned} \Phi_{k,j} &= A^{k-1} A^{k-2} \ldots A^j \qquad k > j \\ \Phi_{j,j} &= I \end{aligned}$$

this satisfies

$$\Phi_{k+1,j} = A^k \Phi_{k,j}$$

$$\Phi_{j,j} = I$$

then we can combine (3b), (3c) and (3d) by solving (3b) by successive substitution. This results in:

(6) $$x^K = \xi^K = \Phi_{K,o}\xi^o + \sum_{j=1}^{K} \Phi_{K,j} B^{j-1} u^{j-1} \ .$$

Then the system (3) ignoring the state space constraints takes on the much simpler form

(7a) Minimize $\sum_{k=0}^{k-1} c_k(u^k) + h^k \left\{ \Phi_{k,o}\xi^o + \sum_{j=1}^{k} \Phi_{k,j} B^{j-1} u^{j-1} \right\}$

(7b) Subject to $\sum_{j=1}^{K} \Phi_{K,j} B^{j-1} u^{j-1} = \xi^K - \Phi_{K,o}\xi^o$

(7c) $E^k u^k \leqq e^k \qquad k = 0,1, \ldots, K-1$.

(7) has $r + rK$ equations in rK variables. As a result of having removed the state space constraints (3f) the constraints of (7) are in a perfect form for Dantzig-Wolfe Decomposition, (Dantzig [7]; Dantzig [9]; Dantzig and Van Slyke [8]) since the constraints on the controls are linked in time only through the relatively small number of equations in (7b). The application of decomposition techniques reduces (7) to a problem which looks more or less like $K + 1$ problems, one with r equations and K with r equations which are solved sequentially instead of (3) which is one big problem with $K[n + m + p] + 2n$ equations (not counting the state constraints) which is solved once.

Now we return to pick up the state space constraints which we have ignored up until now. The idea is the same as in the continuous case.

Algorithm:

Step 0:

Solve (7) for $\hat{u}^1 = (u^{0,1}, \ldots, u^{K-1,1})$, set $\tau = 1$ go to Step 1.

Step 1:

If for $k = 1, \ldots, K$

$$D^k \left\{ \Phi_{k,o} \xi^o + \sum_{j=1}^{k} \Phi_{k,j} B^{k-1} u^{k-1,\tau} \right\} \leqq d^k$$

$\hat{u}^\tau$ is optimal if not set

$$\pi_i^{k,\tau} = 1 \text{ if } D_i^k \left\{ \Phi_{k,o} \xi^o + \sum_{j=1}^{k} B^{k-1} u^{k-1,\tau} \right\} > d_i^k$$

$$= 0 \text{ otherwise}$$

for $i = 1, \ldots, \ell$, $k = 0,1, \ldots, K-1$ where D_i^k and d_i^k are the ith rows of D^k and d^k respectively. Then we add the constraint corresponding to the new state variable $x_{n+\tau}$ given by (8) to Equation (10c) in Step 2 and procede to Step 2

(8a) $$x_{n+\tau}^{k+1} = x_{n+\tau}^k + \pi^{k,\tau}(D^k x^k - d^k) \qquad k = 0,1, \ldots, K-1$$

(8b) $$x_{n+\tau}^o = 0$$

(8c) $$x_{n+\tau}^K \leqq 0 .$$

Equation (8) is equivalent to (9)

(9) $$\sum_{k=1}^{K-1} \pi^{k,\tau} D^k x^k \leqq \sum_{k=1}^{K-1} \pi^{k,\tau} d^k .$$

(9) is added to (10) after substituting for x^k using (6).

Step 2:

Solve

(10a) Minimize $\sum c_k(u^k) + h^k \left\{ \Phi_{k,o}\xi^o + \sum_{j=1}^{k} \Phi_{k,j} B^{j-1} u^{j-1} \right\}$

(10b) Subject to $\sum \Phi_{k,j} B^{j-1} u^{j-1} = \xi^k - \Phi_{k,o}\xi^o$

and

(10c) $\sum_{k=1}^{K-1} \pi^{k,\nu} D^k \left\{ \Phi_{k,o}\xi^o + \sum_{j=1}^{k} \Phi_{k,j} B^{k-1} u^{k-1,\nu} \right\} \leqq \sum_{k=1}^{K-1} \pi^{k,\nu} d^k$ for $\nu = 1, \ldots, \iota$

(10d) $E^k u^k \leqq e^k \qquad k = 0,1, \ldots, K-1$.

The convergence of this algorithm is discussed in Van Slyke and Wets [21]. Moreover if $c_k(u^k)$ uniformly convex in u the redundant inequalities of (10c) can be dropped by Theorem 3.3. If $c_k(u^k)$ is linear for each k the algorithm terminates in a finite number of steps and again redundant constraints may be dropped Van Slyke and Wets [21].

The last point we want to discuss is the validity of the discretization. In particular as the number of discretizations K goes up, do the optimal solutions of the discrete problems approach an optimal solution of the original problem (1)? The answer is yes. The proof of this for autonomous problems is due to Cullum [6]. We need only make slight modifications. The first thing we need are assumptions which guarantee that the discrete problems (3) have solutions for K sufficiently large. We make the following assumption which is somewhat stronger than the one made by Cullum. Let $R = \{z \mid \exists\, x(t), u(t)$ satisfying (1b) - (1f), $z = x(T)\}$. We assume that ξ^T is an interior point of R . Then the results of Cullum apply to our problem. Cullum's proof of this fact depends on the uniformity condition embedded in the following theorem:

Theorem 1:

Let $u^K(t)$ be any uniformly bounded sequence of functions on $[0,T]$ which are constant on each of the intervals $\left[0,\frac{T}{K}\right)$, $\left[\frac{T}{K},\frac{2T}{K}\right)$, ..., $\left[\frac{K-1}{K}T,t\right]$ respectively for $K = 1,2,\ldots$ Let x^{kK}, $k = 0,1,\ldots,K$, $K = 1,2,\ldots$ be the solution of (3b, 3c) corresponding to $u^k = u^K\left(\frac{kT}{K}\right)$. Let $\hat{x}^K(t)$ be the piecewise linear extension of x^{kK}, $k = 0,1,\ldots,K$ to $[0,T]$. Finally, let $x^K(t)$ be the solution of (1b, 1c) with $u(t) = u^K(t)$. Then we have $||\hat{x}^K(t) - x^K(t)|| \to 0$ uniformly for $t \in [0,T]$ as $K \to \infty$.

Remarks:

Cullum established the uniformity result in the case of autonomous systems. Theorem 1 generalizes the result to nonautonomous systems. The proof will be given in the form of several lemmas.

As a preliminary we discuss briefly the norms we will use for matrices. Let $||x||$ be any norm on E^n and let A be an $n \times n$ matrix. Then $||A||^* = \sup \frac{||Ax||}{||x||}$. This norm has all the norm properties, additionally if $C = AB$ then $||C||^* \leqq ||A||^*||B||^*$. Some examples of how this works is

(i) $||X||_2 = \left(\sum X_i^2\right)^{1/2}$ then $||A||_2^* = \sqrt{\lambda}$ where λ is the eigenvalue of A^TA with largest value.

(ii) $||X||_1 = \sum |X_i|$ then $||A||_1^* = \operatorname{Max}_j \sum_i |a_{ij}|$.

(iii) $||X||_\infty = \operatorname{Max}_i |X_i|$ then $||A||_\infty^* = \operatorname{Max}_i \sum_j |a_{ij}|$.

In what follows any pair $||\ ||$, $||\ ||^*$ will be used unless otherwise specified. Note that

$$||X||_\infty \leqq ||X||_1 \leqq n||X||_2$$

and

$$||X||_\infty \leqq ||X||_2 \leqq \sqrt{n}||X||_\infty .$$

To simplify the notation we will drop the super $*$ since it is clear from the context which norm is meant between $|| \;||$ and $|| \;||^*$.

Let $A(t)$ be continuous on $[0,T]$ and consider $\frac{dx}{dt} = A(t)x(t)$ with initial condition $x(0) = \xi^o$. For $K = 1,2, \ldots$ let

$$\frac{x^{k+1} - x^k}{\frac{T}{K}} = A\left(\frac{kT}{A}\right)x^k$$

or

$$x^{k+1} = \left[I + \frac{T}{K} A\left(\frac{kT}{K}\right)\right]x^k$$

with $x^o = \xi^o$. Then by successive substitutions

$$x^k = \Phi_{kj}x^j$$

where for $k > j$

$$\Phi^K_{kj} = \left[I + \frac{T}{K} A\left(\frac{k-1}{K} T\right)\right]\left[I + \frac{T}{K} A\left(\frac{k-2}{K} T\right)\right] \cdots \left[I + \frac{T}{K} A\left(\frac{j}{K} T\right)\right]$$

and $\Phi^K_{jj} = I$. In particular we have

$$x^k = \Phi^K_{k,o}\xi^o .$$

<u>Lemma 1</u>:

$\Phi^K_{k,j}$ is uniformly bounded for $0 \leqq j \leqq k \leqq K$.

<u>Proof</u>:

Since A is continuous $||A(t)|| \leqq \bar{A}$ for some number $\bar{A}$ for all $t \in [0,T]$. Then $\left|\left|\left[I + \frac{T}{K} A\left(\frac{k}{K} T\right)\right]\right|\right| \leqq 1 + \frac{T}{K} \bar{A}$ for all $k = 0,1, \ldots, K$. Thus $||\Phi_{kj}|| \leqq ||1 + \frac{T}{K} \bar{A}||^{k-j} \leqq ||1 + \frac{T}{K} \bar{A}||^K$ for all $0 \leqq j \leqq k \leqq K$. But $||1 + \frac{T}{K} \bar{A}||^K \leqq e^{\bar{A}T}$. Let

$$\phi^K(t;s) = \left[I + (t - kT)A(kT)\right]\left[I + \frac{T}{K}A\left(\frac{k-1}{K}T\right)\right]$$

$$\cdots\left[I + \frac{T}{K}A\left(\frac{j+1}{K}T\right)\right]\left[I + \left(\frac{j+1}{K}T - s\right)A(s)\right]$$

for $\frac{k+1}{K}T \geq t \geq \frac{k}{K}T$ and $\frac{j+1}{K}T \geq s \geq \frac{j}{K}T$. ϕ^K is the piecewise linear continuous extension of $\phi_{k,j}$ to $[0,T]$.

Corollary:

$\phi^K(t;s)$ is uniformly bounded for $0 \leq s \leq t \leq T$.

Lemma 2:

Consider $B = I + A$, where A is an arbitrary fixed matrix. Then there exists λ_o such that $||A|| \leq \lambda_o$ implies B is nonsingular. Moreover for every $\varepsilon > 0$ there exists $\lambda(\varepsilon) > 0$ such that for $||A|| \leq \lambda(\varepsilon)$, $||B^{-1} - I|| \leq \varepsilon$.

Proof:

B is nonsingular if and only if its determinant is nonzero. The determinant of a matrix is a continuous function of its elements. For $||A|| = 0$, $||B|| = 1$ hence there exists λ_o such that $||A|| \leq \lambda_o$ implies $||B|| > 0$. Similarly the inverse of a matrix is a continuous function of its elements wherever it is nonsingular; in particular for $||A|| \leq \lambda_o$. Finally $||B^{-1} - I||$ is a continuous function of the elements of A for $||A|| \leq \lambda_o$. $||B^{-1} - I|| = ||I - I|| = 0$ for $||A|| = 0$ hence for some $\lambda(\varepsilon) > 0$ $||A|| \leq \lambda(\varepsilon)$ implies $||B^{-1} - I|| \leq \varepsilon$.

Lemma 3:

For any $\varepsilon > 0$ there exists $\bar{K}$ such that $K > \bar{K}$ implies $||\frac{d}{dt}\phi^K(t;s) - A(t)\phi^K(t;s)|| \leq \varepsilon$ uniformly for $s \in [0,T]$ and $t \in [0,T] - \left\{0, \frac{T}{K}, \frac{2T}{K}, \ldots, T\right\}$.

<u>Proof</u>:

$$\frac{d}{dt}\,\Phi^K(t;s) = A\left(\frac{k}{K}\,T\right)\left[I + \left(t - \frac{k}{K}\,T\right)A\left(\frac{k}{K}\,T\right)\right]^{-1}\Phi^K(t;s)\ . \quad \left|\left|\left(t - \frac{k}{K}\,T\right)A(kT)\right|\right| \leqq \frac{T}{K}\,\bar{A}$$

hence for any $\varepsilon > 0$ there exists K^1 such that $K > K^1$ implies

$$\left|\left|A\left(\frac{k}{K}\,T\right)\Phi^K(t;s) - A\left(\frac{k}{K}\,T\right)\left[I + \left(t - \frac{k}{K}\,T\right)A\left(\frac{k}{K}\,T\right)\right]^{-1}\Phi^K(t;s)\right|\right|$$

$$\leqq \left|\left|A\left(\frac{k}{K}\,T\right)\left(I - \left[I + \left(t - \frac{k}{K}\,T\right)A\left(\frac{k}{K}\,T\right)\right]^{-1}\right)\Phi^K(t;s)\right|\right|$$

$$\leqq \left|\left|A\left(\frac{k}{K}\,T\right)\right|\right|\ \left|\left|I - \left[I + \left(t - \frac{k}{K}\,T\right)A\left(\frac{k}{K}\,T\right)\right]^{-1}\right|\right|\ \left|\left|\Phi^K(t;s)\right|\right|$$

$$\leqq \frac{\varepsilon}{2}$$

using Lemma 1 and 2 and the fact that $||A(t)||$ is bounded. Since $A(t)$ is continuous on a compact set $[0,T]$ it is uniformly continuous hence there exists K^2 such that $K > K^2$ implies

$$\left|\left|A\left(\frac{k}{K}\,T\right) - A(t)\right|\right| \leqq \frac{\varepsilon}{2\bar{\Phi}} \qquad \text{for } \frac{k+1}{K}\,T \geq t \geq \frac{k}{K}\,T$$

and hence $\left|\left|A\left(\frac{k}{K}\,T\right)\Phi^K(t;s) - A(t)\Phi^K(t;s)\right|\right| \leqq \frac{\varepsilon}{2}$ where $\bar{\Phi}$ is the upper bound on $\Phi^K(t;s)$ given by the corollary to Lemma 1. Thus

$$\left|\left|\frac{d}{dt}\,\Phi^K(t;s) - A(t)\Phi^K(t;s)\right|\right| = \left|\left|A\left(\frac{k}{K}\,T\right)\left[I + \left(t - \frac{k}{K}\,T\right)A\left(\frac{k}{K}\,T\right)\right]^{-1}\Phi^K(t;s)\right.$$

$$\left. - A(t)\Phi^K(t;s)\right|\right|$$

$$\leqq \left|\left|A\left(\frac{k}{K}\,T\right)\left[I + \left(t - \frac{k}{K}\,T\right)A\left(\frac{k}{K}\,T\right)\right]^{-1}\Phi^K(t;s) - A\left(\frac{k}{K}\,T\right)\Phi^K(t;s)\right|\right|$$

$$+ \left|\left|A\left(\frac{k}{K}\,T\right)\Phi^K(t;s) - A(t)\Phi^K(t;s)\right|\right|$$

$$\leqq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\ . \quad ||$$

<u>Lemma 4</u>:

Let $X(x,t)$ be continuous on $E^n \times [0,T]$ and satisfy a Lipshitz condition;

i.e., $||X(x,t) - X(y,t)||_2^* \leq L||x - y||_2$ for all $t \in [0,T]$, and suppose $x(t)$ solves

$$\frac{dx}{dt} = X(x,t)$$

$$x(0) = \xi^o .$$

Suppose $y(t)$ is continuous and has a continuous derivative except at a finite number of simple discontinuities and satisfies

$$y(0) = x(0) = \xi^o$$

$$||\frac{d}{dt} y(t) - X(y(t),t)||_2 \leq \varepsilon$$

for $t \in [0,T]$ except at the discontinuities of $y(t)$ then
$||y(t) - x(t)||_2 \leq \frac{\varepsilon}{L} \{e^{Lt} - 1\} \leq \frac{\varepsilon}{L} \{e^{LT} - 1\}$.

Proof:

Birkhoff and Rota, p. 165, [3].

Lemma 5:

For any $\varepsilon > 0$ there exists $\bar{K}$ such that $K > \bar{K}$ implies
$||\Phi(t;s) - \Phi^k(t;s)|| \leq \varepsilon$ for all $0 \leq s \leq t \leq T$.

Proof:

Let $x(t)$ be the jth column of $\Phi(t;s)$. $x(t)$ solves

$$\frac{dx}{dt} = A(t)x(t)$$

$$x(s) = U^j$$

where

$$U_i^j = (\delta_{ij}) .$$

Let $y^K(t)$ be the jth column of $\Phi^K(t;s)$. $X(x,t) = A(t)x$ is Lipshitz with constant $\bar{A}$. By Lemma 3 for any $\delta > 0$ there exists K^1 such that $K \geq K^1$ implies

$$\delta \geq \left|\left|\frac{d}{dt} \Phi^K(t;s) - A(t)\Phi^K(t;s)\right|\right| \geq \left|\left|\frac{d}{dt} y(t) - A(t)y(t)\right|\right|$$

uniformly in $0 \leq s \leq t \leq T$ except where $\frac{d}{dt} y(t)$ is discontinuous. Thus by Lemma 4, $||y(t) - x(t)||_2 \leq \frac{\delta}{L} \{e^{LT} - 1\}$. If we take

$$\delta < \frac{L\varepsilon}{\nu(e^{LT} - 1)}$$

then

$$||y(t) - x(t)||_2 \leq \frac{\varepsilon}{\nu}$$

or

$$||\Phi(t;s) - \Phi^K(t;s)||^* < \varepsilon \qquad \text{for all } 0 \leq s \leq t \leq T .$$

<u>Lemma 6</u>:

Consider the system

$$\frac{dx}{dt} = A(t)x(t) + B(t)u(t)$$

$$x(0) = \xi^o$$

for any $u(t)$ which is constant on the intervals $\frac{k}{K} T \leq t < \frac{k+1}{K} T$, $k = 0,1, \ldots, K-1$ and is uniformly bounded in norm by $\bar{U}$ (i.e., $||U(t)|| \leq \bar{U}$ for all $t \in [0,T]$) where B is continuous and $||B(t)||^* \leq \bar{B}$ for all $t \in [0,T]$, $A(t)$ is continuous and $||A(t)|| \leq \bar{A}$ for all $t \in [0,T]$. Let $x(t)$ be a solution of this differential equation.

Also consider the approximating system

$$\frac{x^{k+1} - x^k}{\frac{T}{K}} = A\left(\frac{k}{K} T\right)x^k + B\left(\frac{k}{K} T\right)u^k$$

$$x^o = \xi^o$$

where $u^k = u\left(\frac{k}{K} T\right)$. Let x^k , $k = 0,1, \ldots, K$ be the solution.

Then for all $\varepsilon > 0$ there exists $\bar{K}$ such that $K > \bar{K}$ implies

$$\left|\left|x\left(\frac{k}{K} T\right) - x^k\right|\right| \leq \varepsilon \qquad \text{for } k = 0,1, \ldots, K$$

uniformly for all u satisfying the above conditions.

<u>Proof</u>:

$$x\left(\frac{k}{K} T\right) = \Phi\left(\frac{k}{K} T,0\right)\xi^o + \int_0^{\frac{k}{K} T} \Phi\left(\frac{k}{K} T,s\right)B(s)u(s)ds$$

$$= \Phi\left(\frac{k}{K} T,0\right)\xi^o + \sum_{j=0}^{k-1} \int_{\frac{j}{K} T}^{\frac{j+1}{K} T} \Phi\left(\frac{k}{K} T,s\right)B(s)u^j ds \ .$$

On the other hand by successive substitution

$$x^k = \Phi^K_{k,o}\xi^o + \sum_{j=0}^{k-1} \Phi^K_{k,j+1} \frac{T}{K} B\left(\frac{j}{K} T u^j\right)$$

$$= \Phi^K\left(\frac{k}{K} T,0\right)\xi^o + \sum_{j=0}^{k-1} \Phi\left(\frac{k}{K} T,\frac{j+1}{K} T\right)\frac{T}{K} B\left(\frac{j}{K} T\right)u^j \ .$$

Hence

$$\left\|x\left(\frac{k}{K}T\right)-x^k\right\| \leq \left\|\Phi\left(\frac{k}{K}T,0\right)-\Phi^K\left(\frac{k}{K}T,0\right)\right\| \, \|\xi^o\|$$

$$+\left\|\sum_{j=0}^{K-1}\int_{\frac{j}{K}T}^{\frac{j+1}{K}T}\Phi\left(\frac{k}{K}T,s\right)B(s)u^j ds-\sum_{j=0}^{K-1}\int_{\frac{j}{K}T}^{\frac{j+1}{K}T}\Phi^K\left(\frac{k}{K}T,\frac{j+1}{K}T\right)B\left(\frac{j}{K}T\right)u^j ds\right\|$$

$$\leq \left\|\Phi\left(\frac{k}{K}T,0\right)-\Phi^K\left(\frac{k}{K}T,0\right)\right\| \, \|\xi^o\|$$

$$+\sum_{j=0}^{K-1}\|u^j\|\int_{\frac{j}{K}T}^{\frac{j+1}{K}T}\left\|\Phi\left(\frac{k}{K}T,s\right)B(s)-\Phi^K\left(\frac{k}{K}T,\frac{j+1}{K}T\right)B\left(\frac{j}{K}T\right)\right\|ds \; .$$

Immediately by Lemma 5 the first term can be made arbitrarily small. We now turn to the second term. Since Φ^K and B are continuous they are uniformly continuous on $[0,T]$ thus $\left\|\Phi^K\left(\frac{k}{K}T,\frac{j+1}{K}T\right)B\left(\frac{j}{K}T\right)-\Phi^K\left(\frac{k}{K}T,s\right)B(s)\right\|$ can be made uniformly small in j for any $\frac{j}{K}T \leq s \leq \frac{j+1}{K}T$ for K sufficiently large. By Lemma 5

$$\left\|\Phi\left(\frac{k}{K}T,s\right)B(s)-\Phi^K\left(\frac{k}{K}T,s\right)B(s)\right\| \leq \left\|\Phi\left(\frac{k}{K}T,s\right)-\Phi^K\left(\frac{k}{K}T,s\right)\right\|\bar{B}$$

can also be made uniformly small. Thus

$$\left\|\Phi\left(\frac{k}{K}T,s\right)B(s)-\Phi^K\left(\frac{k}{K}T,\frac{j+1}{K}T\right)B\left(\frac{j}{K}T\right)\right\| \leq$$

$$\left\|\Phi\left(\frac{k}{K}T,s\right)B(s)-\Phi^K\left(\frac{k}{K}T,s\right)B(s)\right\|+\left\|\Phi^K\left(\frac{k}{K}T,s\right)B(s)-\Phi^K\left(\frac{k}{K}T,\frac{j+1}{K}T\right)B\left(\frac{j}{K}T\right)\right\|$$

can be made arbitrarily small for K sufficiently large. Since $\|u^j\| \leq \bar{U}$ the result follows.

<u>Proof of Theorem 1:</u>

The theorem follows directly from Lemma 6 and the fact that the $\hat{x}^K(t)$ and $x^K(t)$ are uniformly continuous on $[0,T]$.

The results given here generalize readily to the case where the initial point $x(0)$, and terminal point $x(T)$ are allowed to range over compact convex sets, Dantzig and Van Slyke, Section 4, [8].

7. EXAMPLES

In the previous sections we generate a sequence of controls $\{u^k\}$ such that we can pick subsequences which converge weakly or strongly to the optimal solution of the system under consideration. In general, if the sequence $\{u^k\}$ is an arbitrary minimizing sequence, then it is not always true that it will converge strongly to the optimal control of the system. Conditions for strong convergence were given in Theorem 4.2. Below we give two examples in which arbitrary minimizing sequences only converge weakly, Levitin and Polyak [16].

Example 1:

$$\text{Minimize} \quad \int_0^1 [x(t)]^2 dt$$

$$\text{Subject to} \quad \frac{dx}{dt} = u$$

$$|u(t)| \leq 1 \qquad \text{for all} \quad t \in [0,1]$$

$$x(0) = 0 .$$

The optimal control is $u^* \equiv 0$. The sequence $u^k(t) = \sin kt$ is a minimizing sequence but converges only weakly to the optimal solution $u^* \equiv 0$.

Example 2:

$$\text{Minimize} \quad [x(1)]^2$$

$$\text{Subject to} \quad \frac{dx}{dt} = u$$

$$\int_0^1 [u(t)]^2 dt \leq 1$$

$$x(0) = 0 .$$

Again, the optimal solution is $u^* \equiv 0$. The sequence $u^k(t) = \sin kt$ is a minimizing sequence but converges only weakly to the optimal solution $u^* \equiv 0$.

It is obvious that it can't converge strongly to the optimal solution $u^* \equiv 0$.

Next we give an example of linear optimal control system and apply the algorithm proposed in this paper. We actually solve the problem on the computer by discretization as given in Section 6.

$$\text{Minimize} \quad x_o(2)$$

$$\text{Subject to} \quad \frac{dx_o(t)}{dt} = x_1(t)$$

$$\frac{dx_1(t)}{dt} = u(t)$$

$$T = 2.0$$

$$u(t) \in U(t) \iff |u(t)| \le 1 \qquad \text{for all} \quad t \in [0,2]$$

$$x(t) \in X(t) \iff \begin{bmatrix} 0 & -1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_o(t) \\ x_1(t) \end{bmatrix} \le \begin{bmatrix} t/2 + 0.125 \\ -t/2 + 0.875 \end{bmatrix} \qquad \text{for all} \quad t \in (0,2)$$

$$x(0) = (2,0)$$

$$x(2) = (x_o(2), 0.5) \ .$$

For the above problem, we have

$$A(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad B(t) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\Phi(t,\tau) = \begin{bmatrix} 1 & t-\tau \\ 0 & 1 \end{bmatrix}$$

and hence

$$x(t) = \begin{bmatrix} x_0(t) \\ \\ x_1(t) \end{bmatrix} = \begin{bmatrix} 1 & t \\ \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ \\ 0 \end{bmatrix} + \int_0^t \begin{bmatrix} 1 & t-\tau \\ \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ \\ u(\tau) \end{bmatrix} d\tau \qquad \text{for all} \quad t \in [0,2] \; .$$

If we solve this system without state space constraints, the optimal control we obtain is

$$u(t) \equiv -1 \qquad 0 \leq t \leq 3/4$$

$$u(t) \equiv +1 \qquad 3/4 \leq t \leq 2$$

and the optimal value of the objective function is $1\frac{9}{16}$.

Figure 1 shows the corresponding optimal trajectory and we find that state space constraints are violated. Hence, we define a vector $\pi(t) = (\pi_1(t),\pi_2(t))$ as follows:

$$\pi_1(t) = \begin{cases} 1 & 0.25 \leq t \leq 0.917 \\ \\ 0 & \text{otherwise} \end{cases}$$

$$\pi_2(t) = \begin{cases} 1 & 0.538 \leq t \leq 1.25 \\ \\ 0 & \text{otherwise} \end{cases}$$

and we define a new state co-ordinate as follows:

$$x_2(t) = \int_0^t [\pi_1(t),\pi_2(t)] \begin{bmatrix} -x_1(\tau) - \frac{\tau}{2} - 0.125 \\ \\ -x_1(\tau) + \frac{\tau}{2} - 0.875 \end{bmatrix} d\tau \qquad \text{for all} \quad t \in [0,2]$$

$$x_2(0) = 0 \qquad x_2(2) \leqq 0 \; .$$

We add this variable $x_2(t)$ to the differential system of the problem as mentioned in the algorithm in Section 4. Hence, for the second iteration, we solve the system

$$\text{Minimize} \quad x_o(2)$$

$$\text{Subject to} \quad \frac{dx_o(t)}{dt} = x_1(t)$$

$$\frac{dx_1(t)}{dt} = \quad u(t)$$

$$\frac{dx_2(t)}{dt} = -(\pi_1(t) + \pi_2(t))x_1(t) + \pi_1(t)\left(-\frac{t}{2} - 0.125\right)$$

$$+ \pi_2(t)\left(\frac{t}{2} - 0.875\right)$$

$$x(0) = (2,0,0)$$

$$x(2) = (x_o(2), \tfrac{1}{2}, x_2(2)) \ , \ x_2(2) \leq 0 \ .$$

We again solve this system and in this way, obtain a sequence of optimal solutions as given in Section 4.

Solution by Discretization

Now, we solve the above problem by direct discretization as mentioned in Section 6. We solve the system for two values of number of discretizations.

1) Let $K = 10$, $\Delta = \frac{2}{10} = \frac{1}{5}$

$$A(k\Delta) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad k = 0,1, \ldots, 10$$

$$B(k\Delta) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad k = 0,1, \ldots, 10$$

$$X^k = \left\{ x^k \;\middle|\; \begin{bmatrix} 0 & -1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_o^k \\ x_1^k \end{bmatrix} \leq \begin{bmatrix} \frac{k}{5} + 0.125 \\ -\frac{k}{5} + 0.875 \end{bmatrix} \right\}$$

$$U^k = \left\{ u^k \;\middle|\; \begin{bmatrix} 1 \\ -1 \end{bmatrix} u^k \le \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\} \qquad k = 0,1, \ldots, 9 \; .$$

Define $\Phi[j,k] = [I + \Delta A((k - 1)\Delta)][I + \Delta A((k - 2)\Delta)] \ldots [I + \Delta A(j\Delta)]$ for $j < k$ $k = 0,1, \ldots, K$ and $\Phi[j,j] = I$ for all j . Then

$$\Phi[j,k] = \begin{bmatrix} 1 & 1/5 \\ 0 & 1 \end{bmatrix}^{k-j} \qquad k > j$$

and

$$x^k = \Phi[0,k]x^o + \sum_{j=1}^{k-1} \Phi[j + 1,k]\Delta B(j\Delta)u^j$$

and the linear programming formulation is

$$\text{Minimize} \quad x_o^K$$

$$\text{Subject to} \quad \begin{bmatrix} x_o^K \\ 0 \end{bmatrix} - \sum_{j=0}^{9} \begin{bmatrix} (9 - j)/25 \\ \cdot \\ 1/5 \end{bmatrix} u^j = \begin{bmatrix} 2 \\ -\frac{1}{2} \end{bmatrix}$$

$$- 1 \le u^j \le 1 \qquad j = 0,1,2, \ldots, 9 \; .$$

At the kth iteration, to generate cuts to be added to the above formulation, we generate $\pi = (\pi_1,\pi_2)$ as follows:

$$\pi_1^k = \begin{cases} 1 & \text{if } \displaystyle\sum_{j=0}^{k-1} u^{j(k-1)} + 0.5k + 0.125 < 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\pi_2^k = \begin{cases} 1 & \text{if } \displaystyle\sum_{j=0}^{k-1} u^{j(k-1)} + 0.5k + 4.875 < 0 \\ 0 & \text{otherwise} \end{cases}$$

where $u^{j(k-1)}$ $j = 0,1, \ldots, 9$ is the optimal control at the $(k - 1)$st iteration.

For the next iteration, the new equation to be added to the previous mathematical programming formulation is given by

$$\sum_{k=1}^{10} \begin{bmatrix}\pi_1^k, \pi_2^k\end{bmatrix} \begin{bmatrix} -\sum_{j=0}^{k-1} u^j - 0.5k - 0.125 \\ \\ -\sum_{j=0}^{k-1} u^j + 0.5k - 4.875 \end{bmatrix} \leq 0$$

or

$$\sum_{k=1}^{10} \left(-\pi_1^k, -\pi_2^k\right) \sum_{j=0}^{k-1} u^j \leq \sum_{k=1}^{10} \pi_1^k (0.5k + 0.125)$$

$$+ \sum_{k=1}^{10} \pi_2^k (-0.5k + 4.875) \ .$$

In the above case, we obtain the optimal solution of the discrete problem in four iterations as shown in Figure 2 through 5. The violation of state space constraints at various iterations is shown by crossed lines.

2) Similarly, let $K = 20$ then $\Delta = \frac{1}{10}$. In this case, we have

$$\Phi[j,k] = \begin{bmatrix} 1 & 1/10 \\ \\ 0 & 1 \end{bmatrix}^{k-j} \qquad k > j$$

and the corresponding linear program is

$$\text{Minimize} \quad x_o^K$$

$$\text{Subject to} \quad \begin{bmatrix} x_o^{20} \\ \\ 0 \end{bmatrix} - \sum_{j=0}^{19} \begin{bmatrix} (19 - j)/100 \\ \\ 1/10 \end{bmatrix} u^j = \begin{bmatrix} 2 \\ \\ -\frac{1}{2} \end{bmatrix}$$

$$-1 \leq u^j \leq 1 \qquad j = 0,1,2, \ldots, 19$$

and in order to generate cuts at the kth iteration, we generate $\pi^k = \left(\pi_1^k, \pi_2^k\right)$ $k = 1, 2, \ldots, 19$ as follows:

$$\pi_1^k = \begin{cases} 1 & \text{if } \sum_{j=0}^{k-1} u^{j(k-1)} + 0.5k + 0.75 < 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\pi_2^k = \begin{cases} 1 & \text{if } \sum_{j=0}^{k-1} u^{j(k-1)} - 0.5k + 9.25 \quad 0 \\ 0 & \text{otherwise} \end{cases}$$

and the new equation to be added to the previous mathematical programming formulation is given by

$$\sum_{k=1}^{20} \left(\pi_1^k, \pi_2^k\right) \begin{bmatrix} - \sum_{j=0}^{k-1} u^j - 0.5k - 0.75 \\ - \sum_{j=1}^{k-1} u^j + 0.5k - 9.25 \end{bmatrix} \leqq 0$$

or

$$\sum_{k=1}^{20} \left(-\pi_1^k, -\pi_2^k\right) \sum_{j=0}^{k-1} u^j \leqq \sum_{k=1}^{20} \pi_1^k (0.5k + 0.75)$$

$$+ \sum_{k=1}^{20} \pi_2^k (-0.5k + 9.25) \; .$$

In this case, we obtain optimal solution in six iterations as shown in Figures 6 through 11.

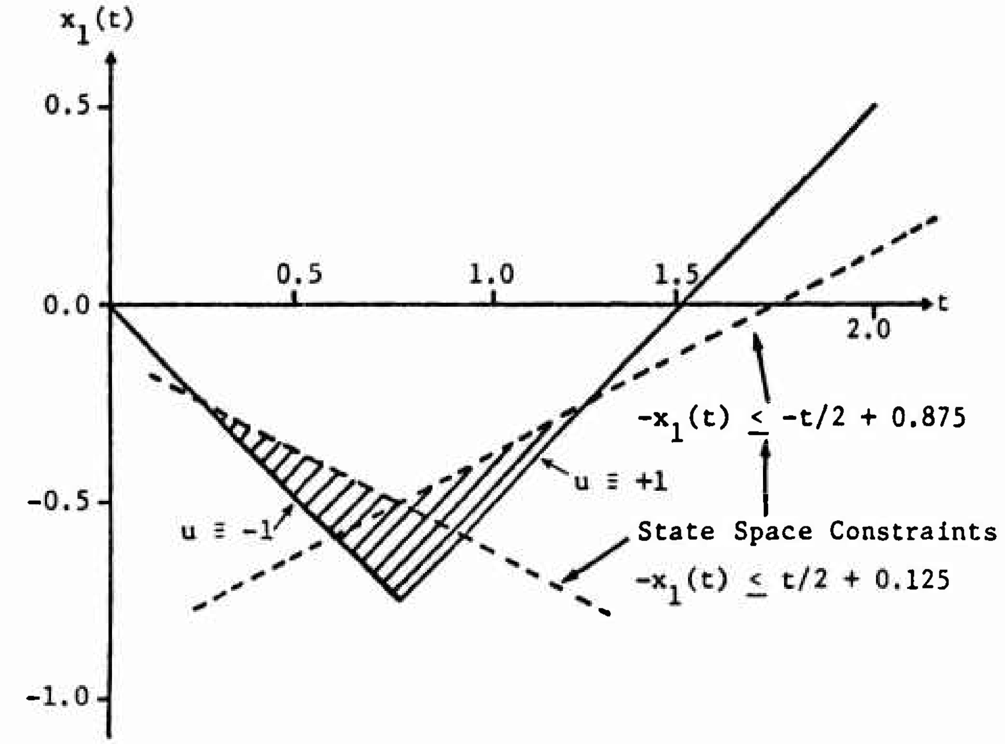


FIGURE 1: OPTIMAL TRAJECTORY WITHOUT STATE SPACE CONSTRAINTS FOR CONTINUOUS SYSTEM

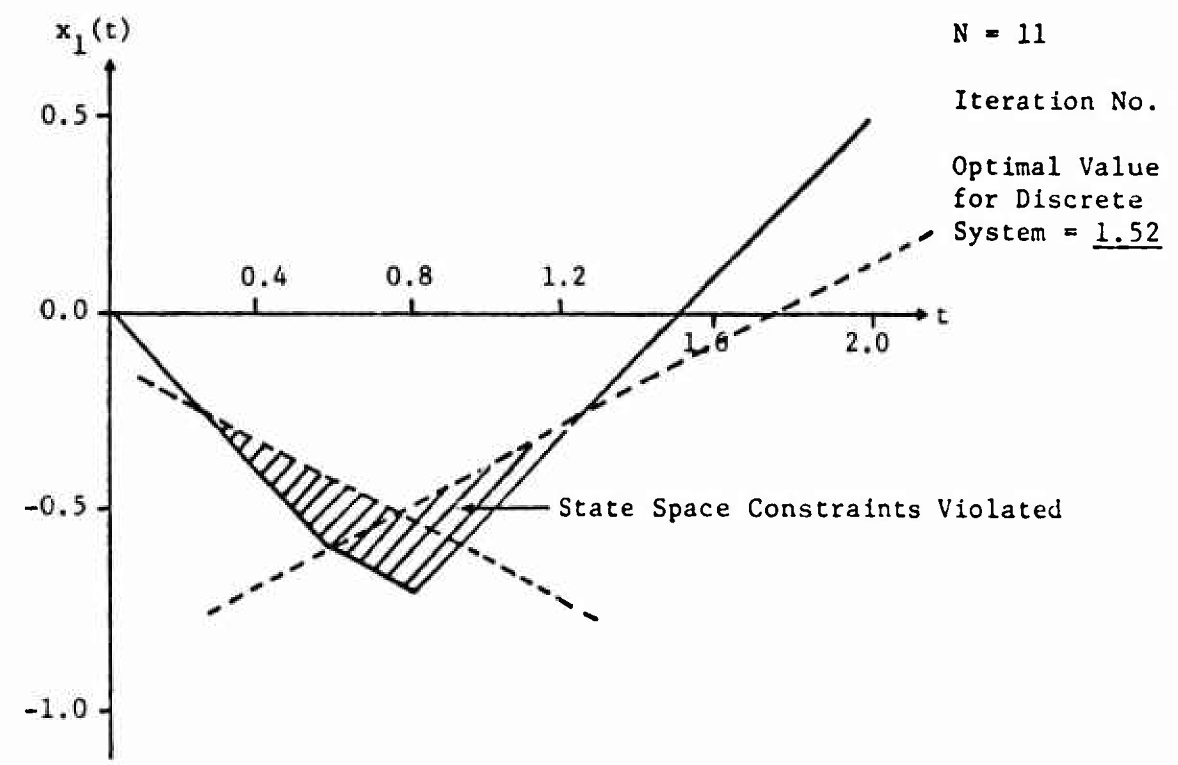
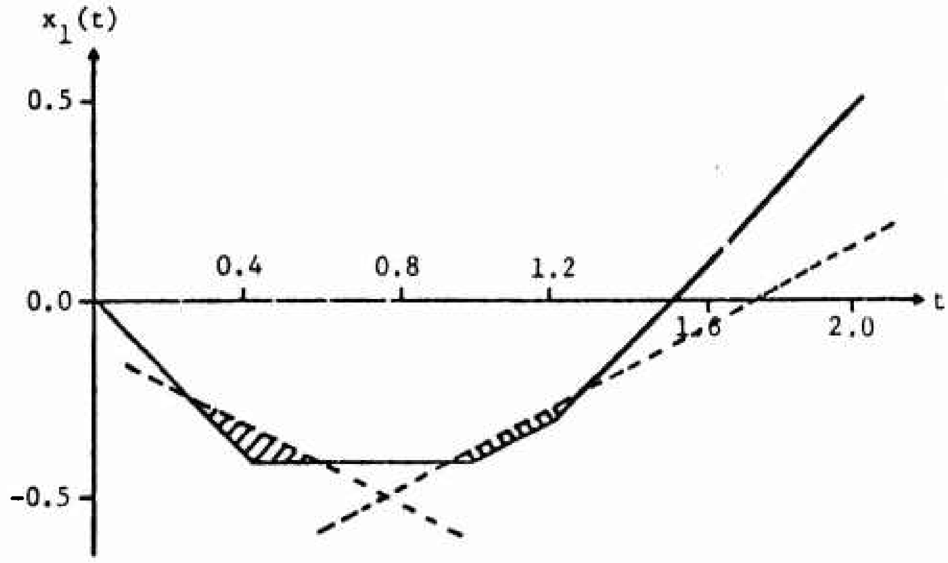


FIGURE 2

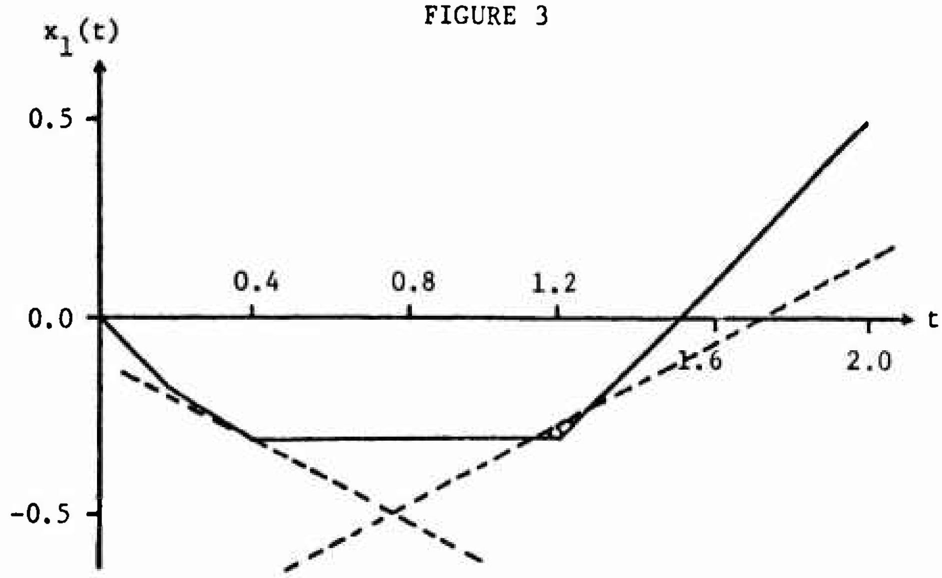


N = 11

Iteration No. 2

Optimal
Value = 1.6336

FIGURE 3

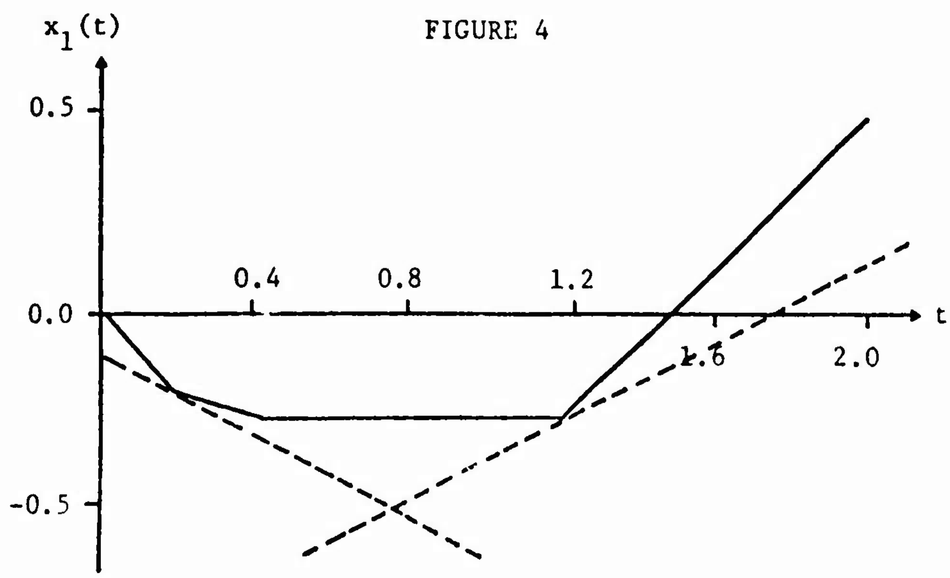


N = 11

Iteration No. 3

Optimal
Value = 1.7050

FIGURE 4



N = 11

Iteration No. 4

Optimal
Value = 1.7450

FIGURE 5: OPTIMAL SOLUTION FOR DISCRETE SYSTEM

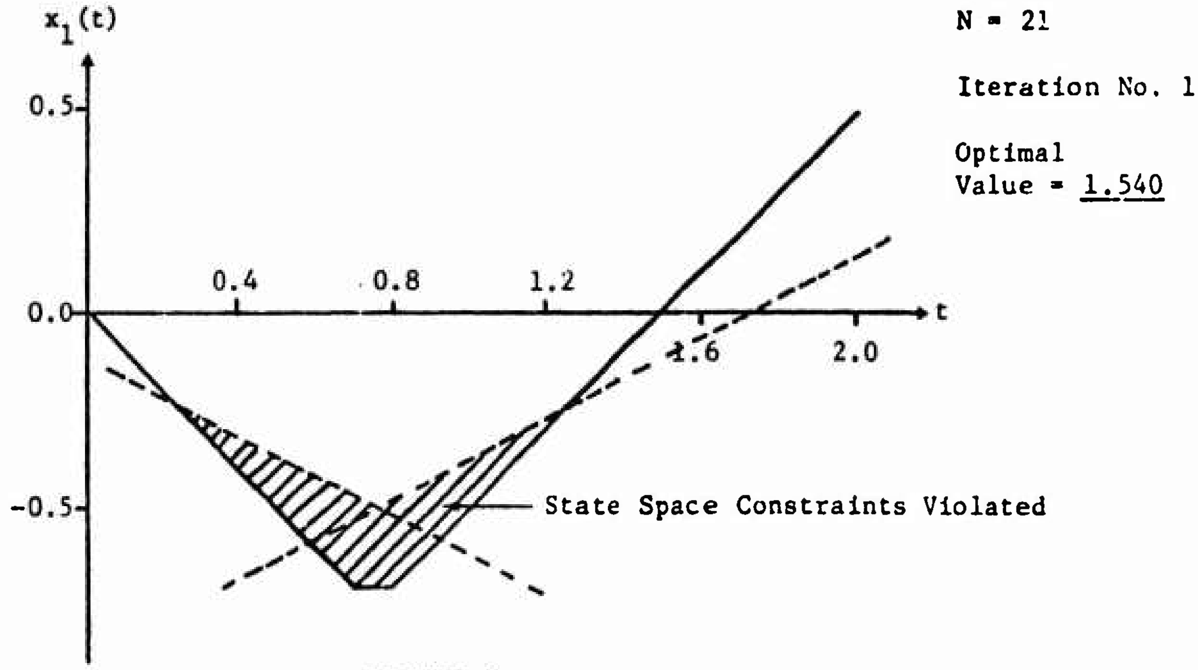


FIGURE 6

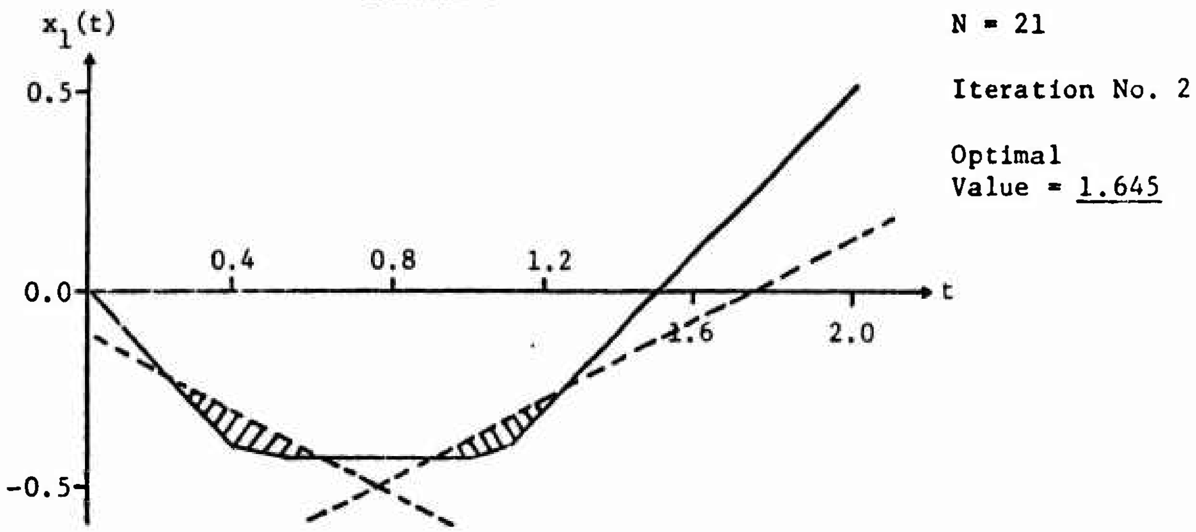


FIGURE 7

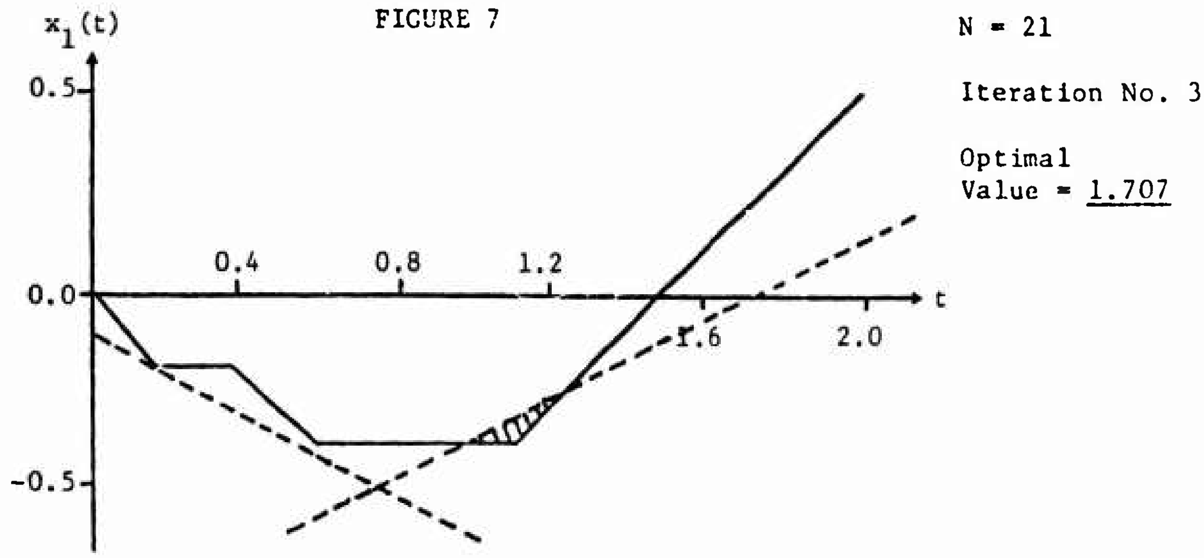
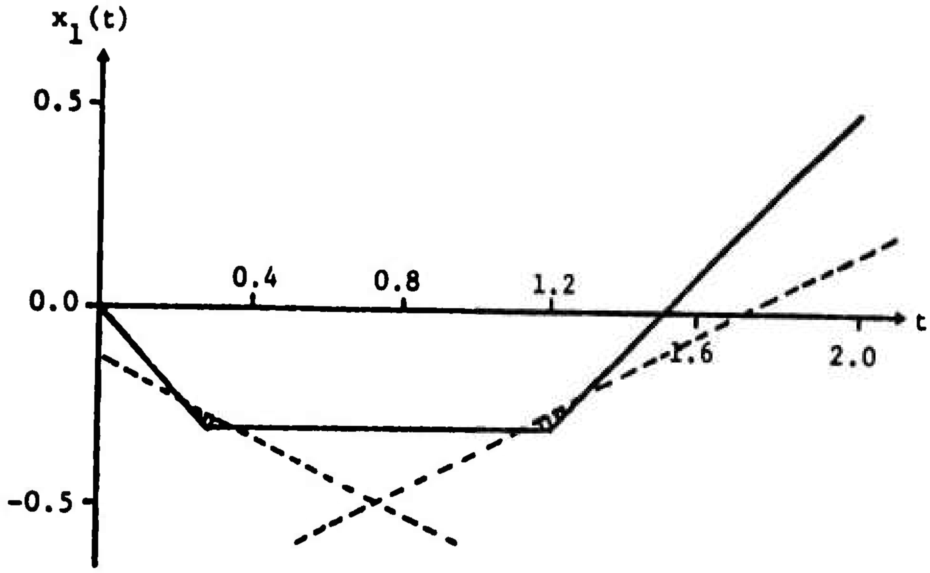


FIGURE 8

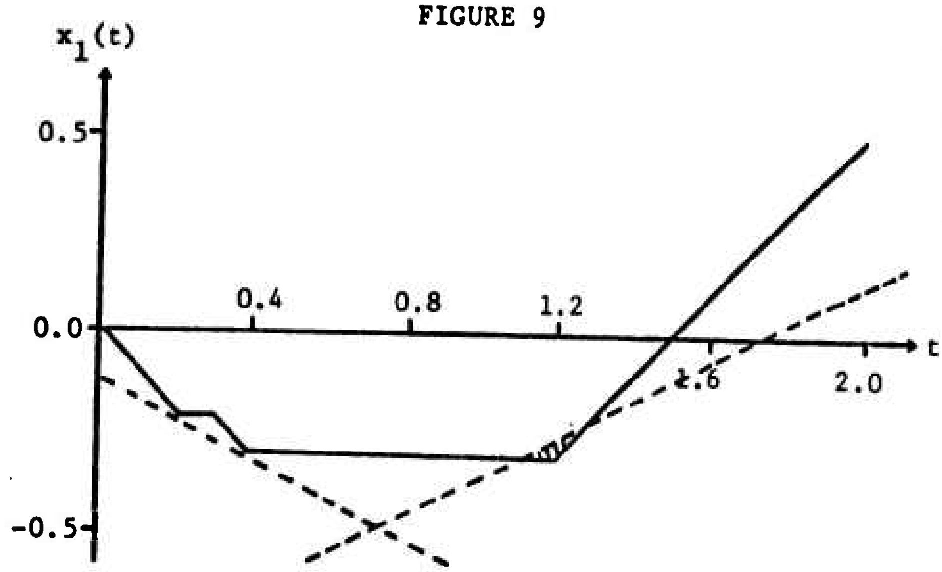


N = 21

Iteration No. 4

Optimal
Value = 1.725

FIGURE 9

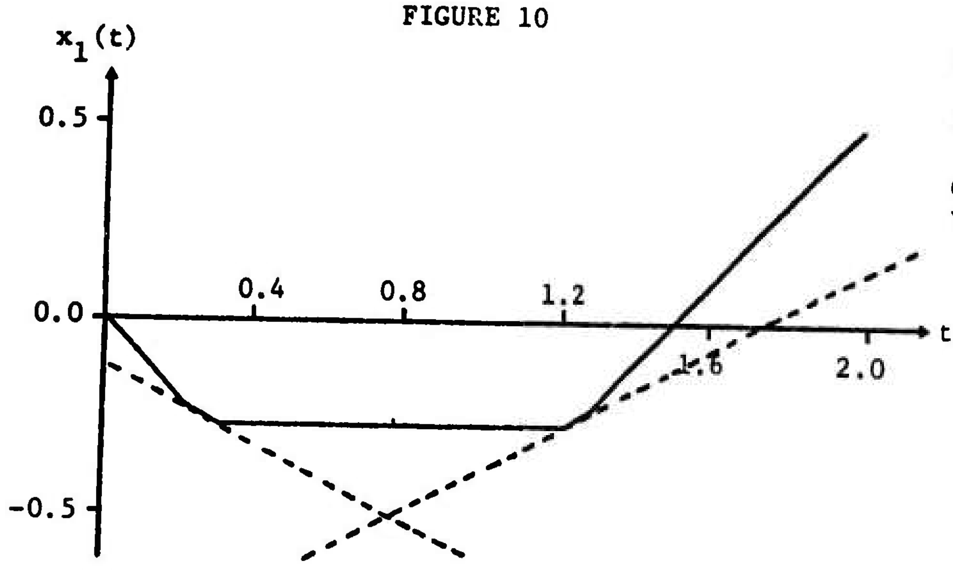


N = 21

Iteration No. 5

Optimal
Value = 1.7344

FIGURE 10



N = 21

Iteration No. 6

Optimal
Value = 1.765

FIGURE 11: OPTIMAL SOLUTION FOR DISCRETE SYSTEM

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Solution by Discretization						

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